Isometric Immersions of Complete Surfaces with Non-positive Curvature

by

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Thesis

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Abstract

In this thesis, we will discuss the problems of isometric immersions of complete surfaces with non-positive Gauss curvature into $\mathbb{R}^3$. We will study Efimov's Theorem which says that any complete surface with Gauss curvature bounded above by some negative constant has no C^2-isometric immersion into $\mathbb{R}^3$. We will also study the result of Hong J. X. on the existence of isometric immersions of complete surfaces with nonpositive curvature. The assumption of Hong's result is basically that the curvature should decay faster than the geodesic distance to the power -2 at infinity. A detailed exposition of the results and their proofs will be given in this thesis. Finally, we will discuss the geometry of the surfaces in Hong's theorem.

摘 要

在此文中，我們將討論具有非正高斯曲率的完備曲面到三維歐氏空間中的等距浸入問題。首先，我們將討論EFIMOV定理：如果一个曲面的高斯曲率小于某一个負數，則此曲面不可以兩次可微的等距浸入到三維歐氏空間中去。然后，我們將研究洪家興的關于存在具有非正高斯曲率的完備曲面的等距浸入的一个結果。洪家興的假設基本上是：在無窮遠處，高斯曲率以快于測地距离的負二次方的速度減小。本文將給這些結果及証明提供詳細的說明。最后，我們還將討論在洪家興的定理中所指的曲面的几何性質。

ACKNOWLEDGMENTS

This thesis is formed by reorganizing the three papers, H-G. Chan [[1], pp. 49-57], Hong J. X. [6] and T. Milnor [7]. More precisely, Chapter 2 follows T. Milnor [7], Theorem 3.1 is due to Hong J. X. [6], and §3.6 follows H-G. Chan [[1], pp. 49-57]. We wish to make their proofs as clearly as possible based on above papers respectively.

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Contents

Chapter 1

Introduction

In this thesis, we will discuss a long standing problem in geometry. Namely, the problem of isometric immersion into $\mathbb{R}^3$ of complete surfaces with non-positive Gauss curvature.

In 1901, D. Hilbert proved that any complete 2-dimensional surface with negative constant Gauss curvature has no C^2 isometric immersion into $\mathbb{R}^3$. In 1964, N. V. Efimov proved

Efimov's Theorem *Any surface cannot be C^2 immersed in $\mathbb{R}^3$ so as to be complete in the induced Riemannian metric, with Gauss curvature $K \leq$ const < 0.*

This is a generalization of Hilbert's theorem.

Efimov's theorem is proved by contradiction. Suppose M is an oriented complete surface with Gauss curvature $K \leq -\kappa < 0$ which can be C^2 isometrically immersed in $\mathbb{R}^3$, where κ is a positive constant, then $M(\mathit{III})$ obtained by using the metric III on M induced by the Gauss map $N : M \longrightarrow \mathbb{S}^2$, has finite area, and N is injective. Using the fact that a complete, simply connected C^2 immersed surface with Gauss curvature $K \leq 0$ has infinite area, and that $|K| \geq \kappa > 0$, one will obtain a contradiction. In Chapter 2, we will give a detailed exposition of the arguments.

Because of Efimov's result, it is natural to try to find sufficient conditions for a complete negatively curved surface to be immersed isometrically in $\mathbb{R}^3$. S. T. Yau has pointed out that a sufficient condition might be related to the rate of decay of the curvature at infinity. The existence question of smooth isometric immersion of a surface into $\mathbb{R}^3$ can be reduced to the construction of solution to the Gauss-Codazzi system. Since the Gauss curvature is negative, Poznjak [8] and Rozhdestvenskii [9] deduced that the Gauss-Codazzi system can be transformed to some equivalent quasi-linear hyperbolic systems. In [6], Hong Jiaxing obtained the following result.

Theorem *(M, g) has a smooth isometric immersion into $\mathbb{R}^3$ if (M, g) with the geodesic coordinates satisfies*

(H_1) $k > 0$ *and* $t\partial_t \ln(k|t|^{2+\delta}) \leq 0$ *as* $|t| \geq T$ *for some positive constants* δ *and* T;

(H_2) $k, \partial_x^i \ln k (i = 1, 2), t\partial_x\partial_t \ln k$ *are bounded, and* $\partial_t^2 \ln k, \partial_t\partial_x \ln k$ *are locally bounded in* t;

(H_3) $\inf\limits_x \int_0^\infty k(x,t)dt$ *and* $\inf\limits_x \int_{-\infty}^0 k(x,t)dt$ *are positive,*

where (M, g) is a complete, simply connected, smooth and 2-dimensional surface with negative curvature $-k$, and g is the Riemannian metric on M, and k is some positive smooth function on M.

We will discuss the theorem in Chapter 3.

Finally, We will also discuss the structure of the class of surfaces studied by Hong [6]. One of the results is related to a conjecture of J. Milnor which says that if S is a complete, umbilic free surface, C^2 immersed in $\mathbb{R}^3$ so that the sum of the squares of the principal curvatures on S is bounded away from zero, then either the Gauss curvature K changes sign on S, or else $K \equiv 0$.

Chapter 2

The Theorem of Efimov

The purpose of the chapter is to prove the Efimov's Theorem. Let M_1^m and M_2^n be at least C^2 manifolds, $n \geq m \geq 1$. An at least C^1 mapping $\varphi : M_1 \longrightarrow M_2$ is called an immersion if $d\varphi_p : T_pM_1 \longrightarrow T_{\varphi(p)}M_2$ is injective for all $p \in M_1$. Fix any surface S (we always assume that it is nonempty, connected 2-manifold, at least C^2.) and a C^2 immersion $i : S \longrightarrow \mathbb{R}^3$, then the induced metric is only C^1. So the intrinsic defintions of Gauss curvature are no longer valid. Define the extrinsic Gauss curvature K by

$$K = (LN - M^2)/(EG - F^2)$$

in terms of the first and second fundamental forms

$$I = Edx^2 + 2Fdxdy + Gdy^2,$$

$$II = Ldx^2 + 2Mdxdy + Ndy^2,$$

of the immersion. It is well known, even if the immersion is only C^2, the extrinsic Gauss curvature K is an invariant of the induced metric I.

Given an at least C^0 Riemannian metric on a surface S, one defines the associated distance $d(p, q)$ between points p and q on S to be the infimum of the lengths of paths in S joining p and q. With the distance so defined, S is called *complete* if it is complete as a metric space, i.e., any d Cauchy sequence in S must be convergent to some point in S.

Theorem 2.1 (Efimov's Theorem) *Any surface can't be C^2 immersed in $\mathbb{R}^3$ so as to be complete in the induced Riemannian metric, with Gauss curvature $K \leq const < 0$.*

2.1 The Idea of the Proof of the Efimov's Theorem

The Efimov's Theorem is proved by contradiction. Suppose a surface M can be C^2 immersed in $\mathbb{R}^3$, with a complete induced metric I, and with extrinsic Gauss curvature $K \leq -\kappa < 0$, where κ is a positive constant. If M is not orientable, one can consider an orientable covering space $\widetilde{M}$ of M. Clearly, if $\widetilde{M}$ can't be C^2 immersed in $\mathbb{R}^3$, then M can't be C^2 immersed in $\mathbb{R}^3$ yet. So one can also suppose that M is oriented.

Let the Gauss map $N : M \longrightarrow \mathbb{S}^2$(unit sphere in $\mathbb{R}^3$), since $K < 0$, then N is a C^1 immersion, inducing the metric $I\!I\!I$ on M. Let $M_{I\!I\!I}$ be the surface M with the metric $I\!I\!I$, then

Lemma 2.1 *$M_{I\!I\!I}$ is not complete.*

Proof Suppose $M_{I\!I\!I}$ is complete, then, pull back the C^∞ structure on $\mathbb{S}^2$ (unit sphere in $\mathbb{R}^3$) to $M_{I\!I\!I}$ by using the C^1 Gauss map $N : M \longrightarrow \mathbb{S}^2$. So, $M_{I\!I\!I}$ is a complete C^∞ manifold with its Gauss curvature equal to 1. By the Bonnet-Hopf-Rinow theorem (cf [[10], VIII, Section 13]), $M_{I\!I\!I}$ is compact. $M_{I\!I\!I}$ is homeomorphic to M, so M is compact. As known, if the immersion is C^2, then the extrinsic Gauss curvature K remains an intrinsic invariant of the induced metric I. Hence, K would be positive at the points on M, whose images in $\mathbb{R}^3$ have maximum distance from the origin. It is impossible. $\square$

Since $M_{I\!I\!I}$ is not complete, one considers the completion $\widetilde{M_{I\!I\!I}}$ of $M_{I\!I\!I}$ as a metric space. The Gauss map N can be extended to a continuous map $\widetilde{N} : \widetilde{M_{I\!I\!I}} \longrightarrow \mathbb{S}^2$.

Definition 2.1 *Let γ be a non geodesic circular arc on $\mathbb{S}^2$, and a point $p \in Int(\gamma)$. Choose the center q of γ, satisfying $\gamma \subset \partial B_q(r)$ and $r \leq \pi/2$. Let γ_σ be the arc obtained by leaving out the distance $\sigma > 0$ from γ along geodesic arcs in the direction away from q on $\mathbb{S}^2$. Let $R(\gamma, \sigma)$ be the rectangle bounded by γ, γ_σ and other two geodesic arcs γ_1, γ_2 (see Figure 2.1), Then $R(\gamma, \sigma)$ is called an exterior rectangle at p.*

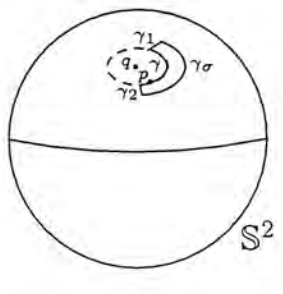

Figure 2.1: $R(\gamma, \sigma)$.

Suppose Ω is any C^1 surface, and map $i : \Omega \longrightarrow \mathbb{S}^2$ is a C^1 immersion, pull back the metric on $\mathbb{S}^2$ to Ω, then Ω has a C^0 Riemannian metric and i is an isometric immersion.

Let $\widetilde{\Omega}$ be the metric completion of Ω, and let $\widetilde{\Omega} \setminus \Omega = \partial\widetilde{\Omega}$. For a set $V \subset \Omega$, let $\overline{V}$ be the closure of V in Ω, and let $\widetilde{V}$ be the closure of V in $\widetilde{\Omega}$. Clearly $\overline{V} = \widetilde{V}$ for a set $V \subset \Omega$ is equivalent to $\widetilde{V} \cap \partial\widetilde{\Omega} = \emptyset$.

Definition 2.2 *Call $\widetilde{\Omega}$ concave at $p \in \partial\widetilde{\Omega}$ if $p \in \widetilde{U}$ of an open set $U \subset \Omega$ such that*

(A) $\widetilde{i}$ is injective on $\widetilde{U}$, and

(B) $i(U)$ contains the interior of an exterior rectangle at $\widetilde{i}(p)$.

Definition 2.3 *Ω is pseudo convex if there exists no $p \in \partial\widetilde{\Omega}$ at which $\widetilde{\Omega}$ is concave.*

Remark 2.1 *Definition 2.2 and Definition 2.3 seem to depend on i, so one should have defined instead the terms " concave at p with respect to i ", or " pseudo convex with respect to i ". But, if Ω and its Riemannian metric are fixed, these notations are independent of the map i. Clearly, $\mathbb{S}^2$ is pseudo convex.*

Example Let φ be the hyperboloid of one sheet given by $x^2 + y^2 - z^2 = 1$. Choose an orientation on φ such that N assigns the " outward " normal at any point $p \in \varphi$. Then N maps φ onto the region $-1/\sqrt{2} < z < 1/\sqrt{2}$ of $\mathbb{S}^2$, and the z coordinate of $N(p)$ on $\mathbb{S}^2$ tends to $1/\sqrt{2}$ as the z coordinate of p on φ tends to $-\infty$, and the z coordinate of $N(p)$ tends to $-1/\sqrt{2}$ as the z coordinate of p on φ tends to ∞. So the metric completion $\bar{\varphi}(\mathit{III})$ is concave at every boundary point.

Definition 2.4 *A nonempty subset H of Ω (or of $\mathbb{S}^2$) is convex if any two points in H can be joined uniquely by a geodesic arc within H whose length equals the distance in Ω (or in $\mathbb{S}^2$, respectively) between the two points. In particular, $\mathbb{S}^2$ is not convex.*

Fix positive constants c and r. Let P denote the parabola $cy^2 = -x$ in $\mathbb{R}^2$, and Let C denote the circle $x^2 + y^2 = r^2$ in $\mathbb{R}^2$. Let

$$D = \{(x, y) \mid 0 < x^2 + y^2 \leq r^2; cy^2 \geq -x\}$$

in $\mathbb{R}^2$, and $\mathcal{D} \subset \mathbb{R}^2$ be any open, simply connected region, such that $D \subset \mathcal{D}$ and $0 \notin \mathcal{D}$(see Figure 2.2).

Suppose $F : \mathcal{D} \longrightarrow \mathbb{R}^2$ is a C^1 immersion, and the metric g^* on $\mathcal{D}$ is the induced metric, so that F is an isometric immersion. Given any two points p and q in D, let $d^*_D(p, q)$ be the infimum of the g^*-lengths of arcs joining p and q in D.

The main result of the proof of the Efimov Theorem is the following so called Efimov's Main Lemma.

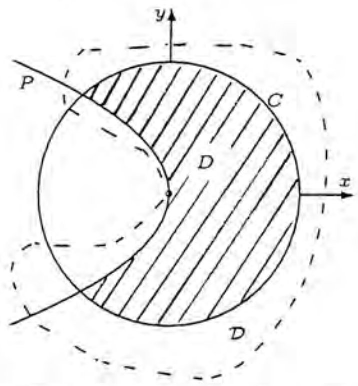

Figure 2.2: D shaded.

Lemma 2.2 (Efimov's Main Lemma) *If $F : \mathcal{D} \longrightarrow \mathbb{R}^2$ is a C^1 immersion, and if the eigenvalues λ_1 and λ_2 of the differential dF throughout $\mathcal{D}$ are real and satisfy*

$$-\alpha \leq \lambda_1 < \lambda_2 \leq \alpha \tag{2.1}$$

for some positive constant α, then D with the distance d_D^* is not a complete metric space.

In terms of Lemma 2.2, the following lemmas will be proved in §2.3 and §2.4, respectively.

Lemma 2.3 *If M is a complete oriented surface C^2 immersed in $\mathbb{R}^3$ with $K \leq -\kappa < 0$, then $M_{\mathbb{I}\mathbb{I}\mathbb{I}}$ is pseudo convex.*

Lemma 2.4 *If $i : \Omega \longrightarrow \mathbb{S}^2$ is a C^1 immersion of a surface Ω, inducing a C^0 Riemannian metric on Ω, such that i is an isometric immersion. If Ω is pseudo convex, then*

(A) i is injective on Ω;

(B) $i(\Omega) = \mathbb{S}^2$, or $i(\Omega)$ is convex (so Ω is simply connected in either case); and

(C) Ω has finite area which is equal to 4π if $i(\Omega) = \mathbb{S}^2$, and is no greater than 2π otherwise.

Efimov's Theorem follows from Lemma 2.3 and Lemma 2.4.

Proof of the Efimov's Theorem Lemma 2.3 and Lemma 2.4 imply that M_{III} is simply connected with finite area no greater than 2π. On the other hand, a complete, simply connected C^2 immersed surface with Gauss curvature $K \leq 0$ has infinite area. So, fix any point p on M, and let $\sigma(r)$ be the area on M of a geodesic disc $D_r(p)$, centered at p, of radius $r > 0$. let $\sigma^*(r)$ be the area using the metric *III* on $D_r(p)$. Then, by Gauss Theorem (cf [[11], p. 156]),

$$2\pi \geq \sigma^*(r) = \int\int_{D_r(p)} |K| d\sigma \geq \kappa \int\int_{D_r(p)} d\sigma = \kappa\sigma(r).$$

But $\sigma(r) \to \infty$ as $r \to \infty$. Hence a contradiction can be obtained and Efimov's generalization holds. $\square$

2.2 Proof of the Efimov's Main Lemma

In this section, we always assume $F : \mathcal{D} \longrightarrow \mathbb{R}^2$ is a C^1 immersion, with $\mathcal{D}$ and D as defined in §2.1, to prove that if $F : \mathcal{D} \longrightarrow \mathbb{R}^2$ is a C^1 immersion, and if the eigenvalues λ_1 and λ_2 of dF throughout $\mathcal{D}$ are real and satisfy

$$-\alpha \leq \lambda_1 < \lambda_2 \leq \alpha$$

for some positive constant α, then D with the distance d_D^* is not a complete metric space.

We first prove the following elementary result.

Lemma 2.5 *If $W \subset \mathbb{R}^+$ is open, and $m(\mathbb{R}^+ \setminus W) < \infty$, where m is the Lebesgue measure on $\mathbb{R}^+$. Then there is no strictly decreasing function $f : W \longrightarrow \mathbb{R}^+$ such that*

$$(u_1 - u)f(u_1) \leq Cf^2(u) \qquad \forall u, u_1 \in W, \tag{2.2}$$

where C is a positive constant.

Proof Let $W \subset \mathbb{R}^+$ be an open set, such that $m(\mathbb{R}^+ \setminus W) < \infty$. Suppose there exists a strictly decreasing function $f : W \longrightarrow \mathbb{R}^+$ satisfying (2.2) for some constant $c > 0$. Let $\psi(u) = -\log f(u), u \in W$, then

$$f(u) = e^{-\psi(u)},$$

function $\psi : W \longrightarrow \mathbb{R}$ is strictly increasing, and

$$(u_1 - u) \leq Ce^{-\psi(u)}(e^{\psi(u_1)-\psi(u)}) \qquad \forall u, u_1 \in W. \tag{2.3}$$

Fix $u \in W$, one obtains from (2.3) that $\psi(u_1) \to \infty$ as $u_1 \to \infty$ in W.
One wants to construct a sequence $x_0 \leq x_1 \leq x_2 \leq ...$ in W, such that for all $i = 0, 1, 2, ...,$

$$\psi(x_{2i+1}) \leq \psi(x_{2i}) + 1 \leq \psi(x_{2i+2}), \tag{2.4}$$

and

$$m(W \cap (x_{2i+1}, x_{2i+2})) < \frac{1}{2^i}. \tag{2.5}$$

The construction is obtained by induction. Take any value in W for x_0. Assume that $x_0 \leq x_1 \leq ... \leq x_{2j}$ have been chosen so that (2.4) and (2.5) hold for $i = 0, 1, ..., (j-1)$, then choose x_{2j+1} and x_{2j+2} as follows.
If $\psi(x_{2j}) + 1 \in \psi(W)$, then define $x_{2j+1} = x_{2j+2} = \psi^{-1}(\psi(x_{2j}) + 1)$.
If $\psi(x_{2j}) + 1 \notin \psi(W)$, then set

$$\sigma_* = \inf\{x \in W \mid \psi(x) > \psi(x_{2j}) + 1\}$$

and

$$\sigma^* = \sup\{x \in W \mid \psi(x) < \psi(x_{2j}) + 1\}.$$

Then $\sigma_* \geq \sigma^*$.
If $x_{2j} < \sigma^*$, take for x_{2j+1} any value in W, such that

$$x_{2j} \leq x_{2j+1} < \sigma^*$$

and

$$\sigma^* - x_{2j+1} < \tfrac{1}{2^{j+1}}.$$

If $x_{2j} = \sigma^*$, set $x_{2j+1} = x_{2j}$. Then take for x_{2j+2} any value in W, such that

$$\sigma_* < x_{2j+2}$$

and

$$x_{2j+2} - \sigma_* < \tfrac{1}{2^{j+1}}.$$

In any case, $x_{2j} \le x_{2j+1} \le x_{2j+2}$ and (2.4) (2.5) hold for $i = j$.
From (2.4), one knows

$$\psi(x_{2i}) \ge \psi(x_0) + i \tag{2.6}$$

for any $i = 0, 1, 2, ...$ So $\psi(x_i) \to \infty$ as $i \to \infty$. Hence, $x_i \to \infty$ as $i \to \infty$. Otherwise, $\{x_i\}$ is bounded by some constant ζ. Note that $\psi(x)$ is strictly increasing, and that $m(\mathbb{R}^+ \setminus W) < \infty$. Then there exists $x \in W$, such that $x > \zeta$ and $\psi(x) < \infty$. So $\{\psi(x_i)\}$ is bounded. It is contradiction with that $\psi(x_i) \to \infty$ as $i \to \infty$. Combining (2.3), (2.4) and (2.6), one obtains

$$x_{2j+1} - x_{2j} \le Ce^{\psi(x_{2i+1}) - 2\psi(x_{2i})} \le Ce^{-\psi(x_{2i})+1} \le Ce^{-\psi(x_0)-i+1}.$$

So

$$\sum_{i=0}^{\infty}(x_{2i+1} - x_{2i}) < \infty.$$

In addition, $(x_{2i+1}, x_{2i+2}) = (W \cap (x_{2i+1}, x_{2i+2})) \cup ((\mathbb{R}^+ \setminus W) \cap (x_{2i+1}, x_{2i+2}))$, and $m(W \cap (x_{2i+1}, x_{2i+2})) < \frac{1}{2^i}$, then

$$x_{2i+2} - x_{2i+1} < \tfrac{1}{2^i} + m((\mathbb{R}^+ \setminus W) \cap (x_{2i+1}, x_{2i+2})).$$

So

$$\sum_{i=0}^{\infty}(x_{2i+2} - x_{2i+1}) < 2 + m(\mathbb{R}^+ \setminus W) < \infty.$$

Hence

$$\sum_{j=0}^{\infty}(x_{j+1}-x_j)<\infty.$$

Thus, $\{x_j\}$ is bounded, which contradicts with the fact $x_j \to \infty$ as $j \to \infty$. Therefore the lemma holds. □

An arc γ in $\mathbb{R}^2$ is given by

$$\gamma(t)=(x(t),y(t)) \qquad t\in[a,b],$$

where $a,b\in\mathbb{R}$ with $b\geq a$, and x,y are two continuous functions on $[a,b]$. If

$$l(\gamma)\overset{\text{def}}{=}\sup\Big\{\sum_{k=1}^{n}|\gamma(t_k)-\gamma(t_{k-1})| \;\Big|\; P=\{t_0=a,t_1,\ldots,t_n=b\}$$
$$\text{is a partition of } [a,b] \text{ with } n\in\mathbb{N}\Big\}<\infty,$$

then γ is rectifiable, and $l(\gamma)$ is just the ordinary Euclidean length of γ. Otherwise, set $l(\gamma)=\infty$ and γ is not rectifiable. For an arc $\gamma\subset\mathcal{D}$, let $l^*(\gamma)$ denote the $g^*-length$ of γ, i.e., $l^*(\gamma)=l(F\circ\gamma)$, and set $l^*(\gamma)=\infty$ if the arc $F\circ\gamma$ in $\mathbb{R}^2$ is not rectifiable. Clearly, $d^*_D(p,q)=\inf\{l^*(\gamma)\mid\gamma \text{ joins } p \text{ to } q \text{ in } D\}$.

Definition 2.5 *A quasi-distance $\tilde{d}$ on D is a function from $D\times D$ to $\mathbb{R}$, satisfying*

(A) $\tilde{d}(p,q)\geq 0$; $\tilde{d}(p,p)=0$;

(B) $\tilde{d}(p,q)=\tilde{d}(q,p)$;

(C) $\tilde{d}(p,q)\leq\tilde{d}(p,r)+\tilde{d}(r,q)$

for any p, q and r in D. clearly, $\tilde{d}(p,q)=0$ is possible even if $p\neq q$. A quasi-distance $\tilde{d}$ is called majorized by d^*_D if for any p, $q\in D$, $\tilde{d}(p,q)\leq d^*_D(p,q)$.

Definition 2.6 *Given a quasi-distance $\tilde{d}$ on D, and an arc $\gamma:[a,b]\longrightarrow D$, set the associated quasi-length*

$$\tilde{l}(\gamma)=\sup\sum_{i=1}^{n}\tilde{d}(\gamma(t_{i-1}),\gamma(t_i))$$

over all finite subdivisions $a = t_0 < t_1 < \ldots < t_n = b$ *of* $[a, b]$. *Clearly,* $\tilde{l}(\gamma)$ *may be infinite. Moreover, the distance* d_D^* *is a quasi-distance, and majorized by itself, the associated quasi-length is* l^*.

Lemma 2.6 *Given a quasi-distance* $\tilde{d}$ *on* D *majorized by* d_D^**, suppose the sequence of arcs* $h_i : [0,1] \longrightarrow D$ *converges uniformly to an arc* $h : [0,1] \longrightarrow D$ *with respect to the* d_D^* *distance, then*

$$\tilde{l}(h) \le \liminf \tilde{l}(h_i).$$

Proof For any positive $s < \tilde{l}(h)$, choose a subdivision

$$0 = x_0 < x_1 < \ldots < x_k = 1$$

of $[0,1]$, such that

$$s < \sum_{j=1}^{k} \tilde{d}(h(x_{j-1}), h(x_j)). \tag{2.7}$$

On the other hand, h_i converges uniformly to h and $\tilde{d}$ is majorized by d_D^*, then for any $\epsilon > 0$, there exists $n = n(s, \epsilon) \in \mathbb{N}$, such that

$$\max_{x\in[0,1]} \tilde{d}(h(x), h_i(x)) \le \max_{x\in[0,1]} d_D^*(h(x), h_i(x)) < \epsilon/k \tag{2.8}$$

for $i \ge n$.

(2.7) and (2.8) imply that

$$\begin{aligned} s &< \sum_{j=1}^{k} \tilde{d}(h(x_{j-1}), h_i(x_{j-1})) + \sum_{j=1}^{k} \tilde{d}(h_i(x_{j-1}), h_i(x_j)) + \sum_{j=1}^{k} \tilde{d}(h_i(x_j), h(x_j)) \\ &< \epsilon + \tilde{l}(h_i) + \epsilon \\ &= 2\epsilon + \tilde{l}(h_i) \end{aligned}$$

for $i \ge n$.

Take supremum on both sides and let $\epsilon \to 0$. then

$$\tilde{l}(h) \le \liminf \tilde{l}(h_i). \quad \square$$

Definition 2.7 *Given $\theta \in \mathbb{R}$, consider the linear function*

$$\xi = x\cos\theta + y\sin\theta$$

defined on $\mathbb{R}^2$. For any rectifiable arc γ in $\mathbb{R}^2$ defined on some interval $[a,b]$, let

$$l_\xi(\gamma) = \sup\Big\{\sum_{k=1}^{n} |\xi(t_k) - \xi(t_{k-1})| \;\Big|\; P = \{t_0 = a, t_1, \ldots, t_n = b\}$$
$$\text{is a partition of } [a,b] \text{ with } n \in \mathbb{N}\Big\},$$

i.e., $l_\xi(\gamma)$ is just the total variation of ξ along γ. So, if $\theta = 0$ (or $\pi/2$), then $l_x(\gamma)$ (or $l_y(\gamma)$, respectively) denotes the total variation of x (or y, respectively) along γ. Moreover,

$$l(\gamma) \leq l_x(\gamma) + l_y(\gamma) \text{ and } l_\xi(\gamma) \leq l(\gamma).$$

If $\gamma \subset \mathcal{D}$, let $l^*_\xi(\gamma)$ denote the total variation of ξ along $F \circ \gamma$, so

$$l^*_\xi(\gamma) = l_\xi(F \circ \gamma).$$

In particular,

$$l^*_x(\gamma) = l_x(F \circ \gamma), \text{ and } l^*_y(\gamma) = l_y(F \circ \gamma).$$

For given any $\theta \in \mathbb{R}$ with $\xi = x\cos\theta + y\sin\theta$, set

$$d^*_\xi(p,q) = \inf\{l^*_\xi(\gamma) \mid \gamma \text{ joins } p \text{ to } q \text{ in } D\},$$

then d^*_ξ is a quasi-distance on D too. Note that $l^*_\xi(\gamma) \leq l^*(\gamma)$ for any arc $\gamma \subset \mathcal{D}$, then d^*_ξ is majorized by d^*_D, and the associated quasi-length is l^*_ξ. So

Corollary 2.1 *Suppose the sequence of arcs $h_i : [0,1] \longrightarrow D$ converges uniformly to an arc $h : [0,1] \longrightarrow D$ with respect to the d^*_D distance, then*

$$l^*_\xi(h) \leq \liminf l^*_\xi(h_i)$$

for any fixed $\theta \in \mathbb{R}$, where $\xi = x\cos\theta + y\sin\theta$.

Definition 2.8 *An arc γ in D joining points p and q is called a g^*-shortest arc γ in D if $l^*(\gamma) = d^*_D(p,q)$, and it is parametrized by its g^*-arclength.*

Remark 2.2 *A g^*-shortest arc γ in D must be C^1 smooth and even if it hits ∂D. In particular, if $\gamma \subset Int(D)$, then it is mapped by F one-to-one onto a line segment in $\mathbb{R}^2$. Moreover, if there exists a point $p \in \partial D \cap Int(\gamma)$, then γ is tangent to ∂D at p. (Cf [[7], Appendix 3].)*

Corollary 2.2 *Suppose the sequence of g^*-shortest arcs $\gamma_i : [a,b] \longrightarrow D$ converges uniformly to an arc $\gamma : [a,b] \longrightarrow D$ in the d^*_D distance, then γ is also a g^*-shortest arc in D.*

Proof First of all, γ is shortest. Since, γ_i are g^*-shortest arcs, so they are parametrized by g^*-arclength. Hence, for any $i \in N$, $d^*_D(\gamma_i(a), \gamma_i(b)) = b - a$. Let $i \to \infty$, then

$$d^*_D(\gamma(a), \gamma(b)) = b - a.$$

In addition, Lemma 2.6 implies

$$l^*(\gamma) \leq \liminf l^*(\gamma_i) = b - a.$$

So $l^*(\gamma) = b - a$.

Then γ is parametrized by g^*-arclength. Since, given $s \in (a,b)$, regard s as b, repeat the above process, then, $l^*(\gamma|_{[a,s]}) = s - a$.

Thus, γ is a g^*-shortest arc. □

Definition 2.9 *An arc $\rho : [0,\infty) \longrightarrow D$ is called a distinguished ray if for any $s > 0$, $\rho|_{[0,s]}$ is a g^*-shortest arc in D. One can see that $\rho(s) \to 0$ as $s \to \infty$. Otherwise, there exists $n \in \mathbb{N}$, such that $\rho \subset (D \cap \{(x,y) \mid x^2 + y^2 \geq 1/n\})$. $D \cap \{(x,y) \mid x^2 + y^2 \geq 1/n\}$ is compact, so the distance between any two points p and q on ρ has the same upper bounded, it is impossible. In addition, if ρ :*

$[0, \infty) \longrightarrow D$, such that $F \circ \rho$ is a Euclidean ray and parametrized by its ordinary arclength, then ρ is a distinguished ray.

Lemma 2.7 *If D is complete with respect to the distance d_D^*, then a distinguished ray exists.*

Proof A distinguished ray is constructed as follows.

Let point $p_0 = (r, 0)$, where $r > 0$ is the radius of the circle C as defined in §2.1. Let

$$\begin{aligned} D(k) &= \{(x,y) \in D \mid x^2 + y^2 \geq r^2/2^k\}, \\ C(k) &= \{(x,y) \in D \mid x^2 + y^2 = r^2/2^k\}, \end{aligned}$$

$k = 1, 2, \ldots$, and let

$$d_{D(k)}^*(p, q) = \inf\{l^*(\gamma) \mid \gamma \text{ joins } p \text{ and } q \text{ in } D(k)\}$$

be the distance on $D(k)$.

For any $k \in \mathbb{N}$, $C(k)$ is compact, so there exists $p_k \in C(k)$, such that,

$$d_{D(k)}^*(p_0, p_k) = d_k \stackrel{\text{def}}{=} d_{D(k)}^*(p_0, C(k)).$$

For each fixed $k \in \mathbb{N}$, there exists a sequence of arcs

$$\gamma_{jk} : [0, l^*(\gamma_{jk})] \longrightarrow D(k)$$

from p_0 to p_k, $j = 1, 2, \ldots$, and each arc is parametrized by its g^*-arclength, such that $l^*(\gamma_{jk}) \to d_k$ as $j \to \infty$. Clearly $l^*(\gamma_{jk}) \geq d_k$ for any $j \in \mathbb{N}$. $D(k)$ is compact, and $\gamma_{jk}|_{[0,d_k]}$ are equicontinuous, by Arzela-Ascoli Theorem, there exists a subsequence of $\gamma_{jk}|_{[0,d_k]}$ converging uniformly to an arc

$$\gamma_k : [0, d_k] \longrightarrow D(k)$$

from p_0 to p_k.

One wants to prove that $\gamma_k : [0, d_k] \longrightarrow D(k)$ is a g^*-shortest arc in $D(k)$ and parametrized by its g^*-arclength.

Obviously, $l^*(\gamma_k) \geq d_k$. On the other hand, by Lemma 2.6,

$$l^*(\gamma_k) \leq \liminf_{j\to\infty} l^*(\gamma_{jk}) = d_k.$$

So $l^*(\gamma_k) = d_k$.

To prove that γ_k is parametrized by its g^*-arclength. Consider $s \in [0, d_k]$, and apply Lemma 2.6 again, then

$$l^*(\gamma_k|_{[0,s]}) \leq \lim_{j\to\infty} l^*(\gamma_{jk}|_{[0,s]}) = s,$$

and

$$l^*(\gamma_k|_{[s,d_k]}) \leq \lim_{j\to\infty} l^*(\gamma_{jk}|_{[s,d_k]}) = d_k - s.$$

Since $l^*(\gamma_k) = d_k$, both inequalities must be equalities. Thus γ_k has a g^*-arclength parameter.

γ_k is in fact g^*-shortest arc in D from p_0 to p_k. Since, suppose a g^*-shortest arc in D from p_0 to p_k, and it goes out of $D(k)$, then it must have some point q at which it hits $C(k)$ first. Note that $0 < d_1 < d_2 < \dots$, let $d_\infty = \lim_{k\to\infty} d_k$, then $d_\infty \leq \infty$. For any $k \geq i \geq 1$, $\gamma_k|_{[0,d_i]}$ is a g^*-shortest arc parametrized by its g^*-arclength, and $\gamma_k \subset D(i)$. Hence $\gamma_k|_{[0,d_i]}$ is equicontinuous. By a diagonal process, there exists subsequence $\{\gamma_{k_j}\}$ of $\{\gamma_k\}$, such that $\gamma_{k_j} \to \rho$ uniformly on any compact subset of $[0, d_\infty)$ as $k_j \to \infty$, where $\rho : [0, d_\infty) \longrightarrow D$ and $\rho|_{[0,s]}$ is a g^*-shortest arc in D for any positive $s < d_\infty$.

One wants to show that ρ is a distinguished ray. To show that $d_\infty = \infty$. Suppose $d_\infty = l^*(\rho) < \infty$, then $\{d_{k_j}\}$ is a Cauchy sequence, and $\rho|_{[0,d_{k_j}]}$ is parametrized by its g^*-arclength for any $i \in \mathbb{N}$, so the points $\rho(d_{k_j})$ form a Cauchy sequence with respect to the d_D^* distance. D is complete, so they converge to a point of D. By the construction, the points $\rho(d_{k_j}) \in C(k_j)$ and hence converge to the origin in $\mathbb{R}^2$ as $k_j \to \infty$, but $0 \notin D$, a contradiction is obtained. So $d_\infty = \infty$.

Thus ρ is a distinguished ray. $\square$

Lemma 2.8 *Suppose γ is a g^*-shortest arc in D, $p \in (D \cap \partial D) \cap Int(\gamma)$, U is a neighborhood of p in D, F is injective on U, and the line L is tangent to $F(\gamma)$ at $F(p)$. Then there exists a closed half disc $\triangle$ in $\mathbb{R}^2$, such that, $\triangle \cap L$ is the diameter and $\triangle \subset F(U)$, moreover, $F(\gamma \cap U) \cap Int(\triangle) = \emptyset$. (See Figure 2.3.)*

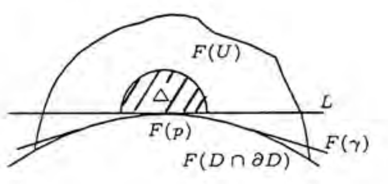

Figure 2.3: $\triangle$ shaded.

Proof Given $p \in (D \cap \partial D) \cap Int(\gamma)$, one can find a neighborhood U of p in D, satisfying that U is simply connected and compact, that $U \cap \partial D$ is connected, that no endpoints of γ are in U, and that F is injective on it.

Remark 2.2 implies that L is tangent to $F(\gamma)$ and $F(U \cap \partial D)$ at $F(p)$, and p is not in a corner of $D \cap \partial D$ (cf [[7], Appendix 3]). Since F is a local diffeomorphism, and $D \cap \partial D$ is smooth, L cuts $\mathbb{R}^2$ at $F(p)$ into two parts consisting of vectors pointing " into " or " out " of $F(U)$. (Otherwise, choose U smaller.) Choose a Euclidean ray starting at $F(p)$, and its initial tangent vector points " into " $F(U)$ at $F(p)$, extend the ray until it hits $\partial F(U)$ at some point q first.

To show that $q \notin F(U \cap \partial D)$. Suppose $q \in F(U \cap \partial D)$. The line segment β joining q and $F(p)$ and the arc of $F(U \cap \partial D)$ form a Jordan curve, let J denote the region bounded by the Jordan curve, let γ^+ be the portion of γ starting at p and pointing " into " J. (See Figure 2.4.) γ^+ must leave U and can't cross $D \cap \partial D$, so $F(\gamma^+)$ must leave $F(U)$ and can't cross $F(D \cap \partial D)$. Hence, $F(\gamma^+)$

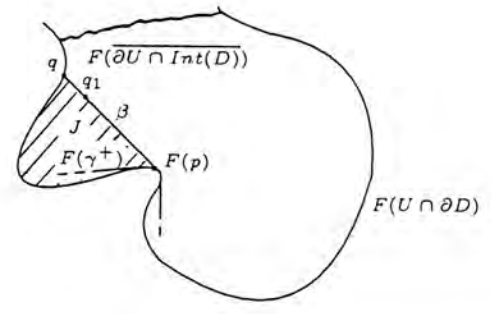

Figure 2.4: J shaded.

must cross β, let q_1 be an intersection point of $F(\gamma^+)$ and β. Then $F(\gamma^+)$ and β have the same subarc joining $F(p)$ and q_1, since F is isometric in U and γ is the g^*-shortest arc. It is impossible since the initial vector of β points " into " $F(U)$.

It implies that $q \in F(\overline{\partial U \cap Int(D)})$. $F(\overline{\partial U \cap Int(D)})$ is compact since $\overline{\partial U \cap Int(D)}$ is a compact subset of ∂U and F is a local diffeomorphism. Suppose the distance between $F(p)$ and $F(\overline{\partial U \cap Int(D)})$ is d, then taking $d/2$ as the radius of the closed half disc $\triangle$ will be required.

To show that $F(\gamma \cap U) \cap Int(\triangle) = \emptyset$. Suppose there exists a point $q_2 \in F(\gamma \cap U) \cap Int(\triangle)$, then the line segment α joining $F(p)$ and q_2 is in $F(U)$. Moreover q_2 is on $F(\gamma)$, since γ is a g^*-shortest arc and F is isometric in U. It is impossible.

Thus the lemma is proved. $\square$

Lemma 2.8 implies the following picture. If ρ is a distinguished ray, then $F \circ \rho$ may look like the curve in Figure 2.5.

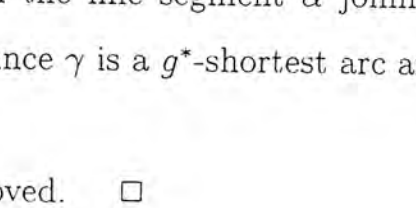

Figure 2.5: $F(D)$ shaded.

Lemma 2.9 *If γ is a g^*-shortest arc in D from a to b and φ is any unself-intersection piecewise C^1 arc in D from a to b. Then for any $\theta \in \mathbb{R}$,*

$$l_\xi^*(\gamma) \leq l_\xi^*(\varphi),$$

where $\xi = x\cos\theta + y\sin\theta$ *as described in Definition* 2.7.

Proof First of all, consider $\theta = 0$, i.e., $\xi = x$.

Case 1 : $\gamma \cap \varphi = \{a, b\}$. Let $B =$ the subset of D bounded by γ and φ, then B is simply connected and compact. Let $G =$ set of all x coordinates at points of $F(\partial B)$ where the tangent lines to $F(\partial B)$ are either undefined or vertical. φ is a piecewise C^1 arc in D, so there are only finitely many points where the tangent lines to $F(\partial B)$ are undefined. If $F(p) = (x^*(p), y^*(p))$ for $p \in D$, then G consists of all critical values of $x^* \circ \gamma$ and $x^* \circ \varphi$, and the values of $x^*(p)$ at the " corners " of ∂B. By Sard's Theorem, $\mathrm{m}(G) = 0$, where m denotes the Lebesgue measure on $\mathbb{R}$. It is easy to see that G is closed.

A parametrized arc l in D is called *prevertical* if $F \circ l$ is a vertical line segment in $\mathbb{R}^2$ and parametrized by the ordinary arclength (it may be open, half open or closed, finite or infinite). So, a prevertical arc l is a g^*-shortest arc in D between any two of its points.

Let $p \in \gamma$, such that $x^*(p) \notin G$. Choose a vertical line segment $\tilde{l}$, such that $\tilde{l}$ points into $F(B)$. Let q be the first point of ∂B such that $l = F^{-1}(\tilde{l})$ meets ∂B. Then $q \notin \gamma$, otherwise, since a prevertical arc is a g^*-shortest arc, then the segment of the g^*-shortest arc γ from p to q would be coincide with l. It is impossible since $p \in \gamma^0$. Let γ^0 be the portion of γ such that $x^*(\gamma^0) \cap G = \emptyset$. Then one can define a map as above

$$\begin{aligned} \Psi : \gamma^0 &\longrightarrow \partial B \\ p &\longrightarrow q(p). \end{aligned}$$

Let $\varphi^0 = q(\gamma^0) \subset \varphi$. Note that $x^*(q(p)) \notin G$ for $p \in \gamma^0$. The mapping is injective because at each point $p \in \partial B$, with $x^*(p) \notin G$, there is only one direction of $F^{-1} \circ \tilde{l}$, where $\tilde{l}$ is a vertical line, such that $F^{-1} \circ \tilde{l}$ points into B.

By construction, one has $l_x^*(\varphi^0) \leq l_x^*(\varphi)$. One wants to prove that $l_x^*(\gamma^0) = l_x^*(\varphi^0)$ and $l_x^*(\gamma) = l_x^*(\gamma^0)$.

To prove $l_x^*(\gamma^0) = l_x^*(\varphi^0)$. Let s be the arclength parametrization of γ^0, and let $p \in \gamma^0$, $q = q(p)$. Then at $F(p)$ and $F(q)$, $F \circ \gamma^0$ and $F \circ \varphi$ can be represented as graphs of $y_1 = y_1(x)$, $y_2 = y_2(x)$ on an interval of x, so $(x, y_2(x))$ is the image of $(x, y_1(x))$ for the map $F \circ \Psi$. But x is a C^1 function of s, so $F \circ \Psi$ and hence φ^0 are C^1 in s. Hence $l_x^*(\gamma^0) = l_x^*(\varphi^0)$.

Next one wants to prove $l_x^*(\gamma) = l_x^*(\gamma^0)$. Let S^0 be the subset of $[0, l^*(\gamma)]$ where

$$\frac{dx^*}{ds}(\gamma(s)) \neq 0.$$

Then

$$l_x^*(\gamma) = \int_{S^0} \left|\frac{dx^*(\gamma(s))}{ds}\right| ds.$$

$x^* \circ \gamma$ is C^1 immersion on S^0 and G is measure zero, so $(x^* \circ \gamma)^{-1}(G) \cap S^0$ has measure zero and hence $l_x^*(\gamma^0) = l_x^*(\gamma)$.

Thus $l_x^*(\gamma) \leq l_x^*(\varphi)$.

Case 2: $(\gamma \cap \varphi) \supsetneq \{a, b\}$. Let $\tilde{\varphi}$ be the portion of φ disjoint from γ

(a) Every point of φ is a point of γ, then $\tilde{\varphi} = \emptyset$. The conclusion of the lemma is true since φ is piecewise C^1.

(b) $\tilde{\varphi} \neq \emptyset$. Since $\tilde{\varphi}$ is open in φ, it can be written by

$$\tilde{\varphi} = \bigcup_i \varphi_i$$

of at most countablely many disjoint open subarcs φ_i of φ, such that

$$l^*(\varphi_1) \geq l^*(\varphi_2) \geq \dots$$

If more than finitely many arcs φ_i are involved, then

$$\lim_{i \to \infty} l^*(\varphi_i) = 0 \qquad (2.9)$$

Make a linear change of parameter if necessary, such that $\varphi : [0,1] \longrightarrow D$. This implies the parametrizations

$$\varphi_i : (a_i, b_i) \longrightarrow D,$$

where the (a_i, b_i) are disjoint subintervals of $[0,1]$. Consider the subarc γ_1 of γ between the endpoints $\varphi(a_i)$ and $\varphi(b_i)$ of φ_1. Making a linear change in the g^*-arclength parametrization of γ if necessary, one has

$$\gamma_1 : [a_1, b_1] \longrightarrow D$$

with $\gamma_1(a_1) = \varphi(a_1)$ and $\gamma_1(b_1) = \varphi(b_1)$. Since γ_1 and the closure of the arc φ_1 meet only at their endpoints, so

$$l_x^*(\varphi_1) \geq l_x^*(\gamma_1).$$

Define a new arc $l_1 : [0,1] \longrightarrow D$ by

$$l_1(t) = \begin{cases} \varphi(t) & \text{if } t \notin [a_1, b_1] \\ \gamma_1(t) & \text{if } t \in [a_1, b_1], \end{cases}$$

one has $l_x^*(\varphi) \geq l_x^*(l_1)$.

Similarly, the arc γ_2 of γ between the endpoints $\varphi(a_2)$ and $\varphi(b_2)$ of φ_2 may be reparametrized if necessary in order to become

$$\gamma_2 : [a_2, b_2] \longrightarrow D$$

with $\gamma_2(a_2) = \varphi(a_2)$ and $\gamma_2(b_2) = \varphi(b_2)$, so

$$l_x^*(\varphi_2) \geq l_x^*(\gamma_2).$$

Define a new arc $l_2 : [0,1] \longrightarrow D$ by

$$l_2(t) = \begin{cases} l_1(t) & \text{if } t \notin [a_2, b_2] \\ \gamma_2(t) & \text{if } t \in [a_2, b_2], \end{cases}$$

then $l_x^*(\varphi) \geq l_x^*(l_1) \geq l_x^*(l_2)$.

Continuing this procedure, one has a sequence of arcs $l_j : [0,1] \longrightarrow D$ each joining $\varphi(0)$ to $\varphi(1)$. If there are only m subarcs φ_i, where $1 \leq m < \infty$, then set $l_j = l_m$ for every $j \geq m$. So

$$l_x^*(\varphi) \geq l_x^*(l_1) \geq l_x^*(l_2) \geq \ldots$$

with all the arcs $l_j \subset (\gamma \cap \varphi)$.

(2.9) implies that for any fixed $\epsilon > 0$, there exists $n \in \mathbb{N}$, such that $l_x^*(\varphi_i) < \epsilon$ for all $i \geq n$. On the other hand, if $j \geq k \geq n$, then

$$\sup_{t \in [0,1]} d_D^*(l_j(t), l_k(t)) \leq 2l^*(\varphi_k),$$

since $l^*(\varphi_1) \geq l^*(\varphi_2) \geq \ldots$. Hence the arcs l_j converge uniformly in the d_D^* distance to a continuous arc $l : [0,1] \longrightarrow D$, by the Corollary 2.1, one obtains

$$l_x^*(\varphi) \geq \lim_{j \to \infty} l_x^*(l_j) \geq l_x^*(l). \tag{2.10}$$

By the construction, all points of l are on γ, and l is continuous, so by (a), one has $l_x^*(l) \geq l_x^*(\gamma)$, using (2.10), one obtains

$$l_x^*(\gamma) \leq l_x^*(\varphi).$$

If $\theta \neq 0$, one can prove similarly by a rotation in the target, the conclusion is also true.

Hence, the lemma holds. □

Remark 2.3 *The conclusion of the above lemma is also true if the piecewise C^1 arc φ has finitely many self-intersections, since one can use the new piecewise C^1 arc $\widetilde{\varphi} \subset \varphi$ joining the original endpoints of φ, without self-intersection, to obtain $l_\xi^*(\gamma) \leq l_\xi^*(\widetilde{\varphi}) \leq l_\xi^*(\varphi)$.*

In the remainder of §2.2, we always assume the eigenvalues λ_1 and λ_2 of dF are real, and satisfy (2.1) over $\mathcal{D}$ for some constant α, with the assumption given in the beginning of the section. F is C^1, so dF is C^0, moreover dF can be written with respect to standard basis of the domain and target by a matrix $\mathbf{A}$, such that

$$dF = \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where a_{ij}, $i, j = 1, 2$, are C^0 functions in $\mathcal{D}$.

By the assumption, the two eigenvalues are real, distinct and non-zero, and $\lambda_2 > \lambda_1$, so if (a_2, b_2) is the eigenvector corresponding to λ_2, then

$$\begin{pmatrix} a_{11} - \lambda_2 & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = 0,$$

i.e.,

$$\begin{cases} (a_{11} - \lambda_2)a_2 + a_{12}b_2 = 0, \\ a_{21}a_2 + (a_{22} - \lambda_2)b_2 = 0. \end{cases} \tag{2.11}$$

The eigenspace of λ_2 is 1-dimension, so

$$\det \begin{pmatrix} a_{11} - \lambda_2 & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{pmatrix} = 0.$$

Hence the system (2.11) is equivalent to one of the two equations in (2.11). Moreover $(a_{11} - \lambda_2)^2 + a_{12}^2$ and $a_{21}^2 + (a_{12} - \lambda_2)^2$ can not be zero at the same time. Otherwise, $a_{12} = a_{21} = 0$, and $a_{11} = a_{22}$, then $\lambda_1 = \lambda_2$, it is impossible. Hence let $|a_2| = |\frac{-a_{12}}{((a_{11}-\lambda_2)^2+a_{12}^2)^{1/2}}|$, $|b_2| = |\frac{a_{11}-\lambda_2}{((a_{11}-\lambda_2)^2+a_{12}^2)^{1/2}}|$ if $(a_{11} - \lambda_2)^2 + a_{12}^2 \neq 0$, or let $|a_2| = |\frac{(a_{22}-\lambda_2)}{(a_{21}^2+(a_{12}-\lambda_2)^2)^{1/2}}|$, $|b_2| = |\frac{a_{21}}{(a_{21}^2+(a_{12}-\lambda_2)^2)^{1/2}}|$ if $a_{21}^2 + (a_{12} - \lambda_2)^2 \neq 0$. One can choose suitable sign for a_2 and b_2, such that a_2 and b_2 are C^0 in $\mathcal{D}$.

Similarly, one can obtain another eigendirection (a_1, b_1) corresponding to λ_1, and a_1, b_1 are C^0 in $\mathcal{D}$.

(a_1, b_1) and (a_2, b_2) are called the first and the second *eigendirection* in $\mathcal{D}$, respectively. So one can see that the two eigendirections are C^0, and are not singular, moreover they are never equal to each other in $\mathcal{D}$ since $\lambda_1 \neq \lambda_2$ everywhere.

A C^1 arc in $\mathcal{D}$ which is tangent to the first (or second) eigendirection everywhere is called a first (or second, respectively) *eigenarc*. Clearly, the tangent vector to an eigenarc γ at q in $\mathcal{D}$ is mapped by dF to a parallel vector tangent to $F \circ \gamma$ at $F(q)$. And dF will preserve the orientation of the tangent vector if the associated eigenvalue is positive at q, and reverse the orientation if the eigenvalue is negative at q.

D is simply connected, so one can fix an orientation on each eigendirection field, obtain two C^0 fields of unit vectors in D. By the Peono Existence Theorem (cf [[5], p. 10]), one obtains that through any point in D, there exist at least one first eigenarcs and at least one second eigenarcs. D is simply connected, and the eigendirections are C^0 and singularity free, so the eigenarcs can't intersect themselves, respectively, hence there exist no closed eigenarcs (cf [[5], p. 150]).

Definition 2.10 *Given any fixed* $\theta \in \mathbb{R}$ *and the associated linear function* $\xi(x, y) = x\cos\theta + y\sin\theta$, *a piecewise* C^1 *parametrized arc in* D *is called a* ξ*-chain if it consists of finitely many eigenarcs and along which* ξ *is strictly increasing.*

Lemma 2.10 *If* $p \in Int(D)$, *and* $\xi = x\cos\theta + y\sin\theta \geq 0$ *at* p, *then there exists a* ξ*-chain* φ *which starts at* p *and ends at a point on* ∂D.

Proof Given θ, consider the coordinates ξ, η in $\mathbb{R}^2$, where

$$\begin{cases} \xi = x\cos\theta + y\sin\theta, \\ \eta = -x\sin\theta + y\cos\theta. \end{cases}$$

So

$$\begin{cases} x = \xi\cos\theta - \eta\sin\theta, \\ y = \xi\sin\theta + \eta\cos\theta. \end{cases} \tag{2.12}$$

Let W_1 (or W_2) be the subset of D consisting of all points where the first (or second) eigendirection is parallel to the η-axis. a_1, b_1, a_2 and b_2 are C^0, so W_1 (or W_2) is closed in D. Moreover $W_1 \cap W_2 = \emptyset$ since the two eigendirections never coincide in D.

Suppose $v_1 = (x(t), y(t))$ is a first eigenarc where t is a parameter, then

$$\begin{cases} \frac{dx}{dt} = a_1(x, y), \\ \frac{dy}{dt} = b_1(x, y). \end{cases}$$

So

$$\begin{cases} \frac{d\xi}{dt} = a_1(x,y)\cos\theta + b_1(x,y)\sin\theta, \\ \frac{d\eta}{dt} = -a_1(x,y)\sin\theta + b_1(x,y)\cos\theta. \end{cases}$$

If $p \in D \setminus W_1$, then $\frac{d\xi}{dt} = a_1(x,y)\cos\theta + b_1(x,y)\sin\theta \neq 0$ at p. Hence

$$\frac{d\eta}{d\xi} = \frac{-a_1(x,y)\sin\theta + b_1(x,y)\cos\theta}{a_1(x,y)\cos\theta + b_1(x,y)\sin\theta}$$

along v_1 in $D \setminus W_1$. By (2.12), x, y are two functions of ξ and η, let

$$f_1(\xi,\eta) = \frac{-a_1(x,y)\sin\theta + b_1(x,y)\cos\theta}{a_1(x,y)\cos\theta + b_1(x,y)\sin\theta},$$

then

$$\frac{d\eta}{d\xi} = f_1(\xi,\eta)$$

where $f_1(\xi,\eta)$ is the C^0 function of ξ and η in $D \setminus W_1$.

Similarly, if v_2 is a second eigenarc, then

$$\frac{d\eta}{d\xi} = f_2(\xi,\eta)$$

along v_2 for some C^0 function f_2 of ξ and η in $D \setminus W_2$.

Thus, an eigenarc nowhere parallel to the η-axis can be regarded as the graph of a C^1 function $\eta = \eta(\xi)$, satisfying

$$\frac{d\eta}{d\xi} = f_1(\xi,\eta)$$

or

$$\frac{d\eta}{d\xi} = f_2(\xi,\eta).$$

The inverse is also true, i.e., if a curve $\eta = \eta(\xi)$ in $D \setminus W_1$, satisfying

$$\frac{d\eta}{d\xi} = f_1(\xi,\eta),$$

then the curve must be a first eigenarc, and it is similar to a second eigenarc.

Given $p \in Int(D)$, at which the ξ coordinate is no less than 0. Let $\xi_0 = \xi$ coordinate at p, then $\xi_0 \geq 0$. Let $H = \{(x, y) \in \mathbb{R}^2 \mid \xi \geq \xi_0\}$, then $W_1 \cap H$ and $W_2 \cap H$ are precompact in $\mathcal{D}$. Let $d = d(W_1 \cap H, W_2 \cap H)$, and let

$$\begin{aligned} N(W_1 \cap H) &= \{(\xi, \eta) \in D \mid d(q_1, W_1 \cap H) \leq d/3, \text{ where } q_1 = (\xi, \eta)\}, \\ N(W_2 \cap H) &= \{(\xi, \eta) \in D \mid d(q, W_2 \cap H) \leq d/3, \text{ where } q = (\xi, \eta)\}. \end{aligned}$$

Then

$$d(N(W_1 \cap H), N(W_2 \cap H)) \geq d/3.$$

So, $N(W_1 \cap H) \cap N(W_2 \cap H) = \emptyset$. Hence, one can assume that $p \notin N(W_1 \cap H)$.

To show that there exists a ξ-chain starting at p and ending at ∂D if $\xi_0 > 0$. By assumption, $p \notin N(W_1 \cap H)$, so near p, the first eigenarc starting at p can be represented by

$$d\eta/d\xi = f_1(\xi, \eta)$$

over the maximal interval $[\xi_0, \xi_1)$ with $\eta(\xi_0)$ equal to the η coordinate at p. Note that the graph of the solution must be in $D \setminus W_1$. If it doesn't hit $N(W_1 \cap H)$, then the graph is in the compact set

$$\overline{(D \cap H) \setminus N(W_1 \cap H)},$$

and the function $f_1(\xi, \eta)$ is bounded on it. So $\eta(\xi)$ will be tend to a limit $\eta_1 < \infty$ as $\xi \to \xi_1$. Moreover $(\xi_1, \eta_1) \in \partial D$, otherwise, $(\xi_1, \eta_1) \in Int(D)$, then the solution $\eta = \eta(\xi)$ can be extended to the larger interval $[\xi_0, \xi_1 + \sigma)$ for some positive constant σ.

If it hits $N(W_1 \cap H)$, say at q, then $q \notin N(W_2 \cap H)$. Similarly, one can obtain a second eigenarc starting at q with ξ increasing along the arc, and hitting ∂D or $N(W_2 \cap H)$.

Continuing this procedure finitely many steps, the last eigenarc must end at a point of ∂D. Otherwise, the arc φ consisting of the eigenarcs would have infinitely ordinary Euclidean length, since $d(N(W_1 \cap H), N(W_2 \cap H)) \geq d/3$. It is impossible, since ξ is bounded in D, and $f_1(\xi, \eta), f_2(\xi, \eta)$ are bounded in $D \setminus W_1$ and $D \setminus W_2$, respectively, which implies that the length of φ is finite.

If $\xi_0 = 0$, then there exists some sufficient small portion of a first eigenarc which leaves p with ξ increasing, since $p \notin N(W_1 \cap H)$, so the first eigendirection can't parallel to η-axis near p. Pick up a point p' close to p, then $p' \notin N(W_1 \cap H)$ and $\xi > 0$ at p'. Using the above conclusion, there is a ξ-chain starting at p' and ending at ∂D.

Thus the lemma holds. □

Lemma 2.11 *For any ξ-chain φ,*

$$l^*_\xi(\varphi) \leq \alpha l_\xi(\varphi) \tag{2.13}$$

where α is the constant in the Main Lemma.

Proof It is sufficient to show that, for any subeigenarc φ_i of φ, $i = 1, 2, ..., k$, where k is the number of the eigenarcs of φ,

$$l^*_\xi(\varphi_i) \leq \alpha l_\xi(\varphi_i).$$

Since φ_i is C^1, and can be written by $\varphi_i(t) = (x_i(t), y_i(t)), t \in [a_i, b_i]$ for some constants a_i, b_i and some C^1 functions x_i, y_i on $[a_i, b_i]$, then $F \circ \varphi_i(t) = (\tilde{x}_i(t), \tilde{y}_i(t))$, for some C^1 functions $\tilde{x}_i, \tilde{y}_i$ on $[a_i, b_i]$. φ_i is a eigenarc, so

$$(\tilde{x}'_i(t), \tilde{y}'_i(t)) = dF(x'_i(t), y'_i(t)) = \lambda_j(x'_i(t), y'_i(t)),$$

where $j = 1$ or 2. Hence

$$\begin{aligned} l_\xi^*(\varphi_i) &= \int_{a_i}^{b_i} |\tilde{x}_i'(t)\cos\theta + \tilde{y}_i'(t)\sin\theta| dt \\ &= \int_{a_i}^{b_i} |<(\tilde{x}_i'(t), \tilde{y}_i'(t)), (\cos\theta, \sin\theta)>| dt \\ &= \int_{a_i}^{b_i} |\lambda_j <(x_i'(t), y_i'(t)), (\cos\theta, \sin\theta)>| dt, j = 1, \text{or } 2. \\ &\leq \alpha \int_{a_i}^{b_i} |<(x_i'(t), y_i'(t)), (\cos\theta, \sin\theta)>| dt \\ &= \alpha l_\xi(\varphi_i). \end{aligned}$$

Thus the lemma holds. □

Note that $|\xi| \leq r$ in D, then

Corollary 2.3 *If φ is a ξ-chain, then*

$$l_\xi^*(\varphi) \leq 2\alpha r, \tag{2.14}$$

where r is the radius of the circle C along ∂D.

Lemma 2.12 *If ρ is a distinguished ray in D, then $l_x^*(\rho) < \infty$, and hence, $l_y^*(\rho)$ is infinite.*

Proof Let $\rho(t)$, $0 \leq t < \infty$, be a distinguished ray, where $t = g^*$-arclength parameter, let $p_0 = \rho(0)$, and let l be a vertical line segment in D from p_0 to a point $q_0 \in C_0 = C \cap D$, where C is the circle $x^2 + y^2 = r^2$. If $p_0 \in C_0$, then $q_0 = p_0$. Then there are two possibilities:

Case 1: There is a sequence $t_j \to \infty$ of g^*-arclength parameter values of ρ, such that $x(\rho(t_j)) \geq 0$; or

Case 2: There is some $\delta \geq 0$, $x(\rho(t)) < 0$ for every $t > \delta$.

Suppose the first case is true. Let $p_j = \rho(t_j)$, and let $\rho_j = \rho|_{[0,t_j]}$. Since $x(p_j) \geq 0$, by Lemma 2.10, there exists a x-chain φ_j joining p_0 to a point $q_j \in \partial D$,

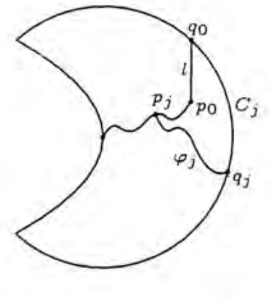

Figure 2.6: Case 1.

with x increasing along φ_j. (See Figure 2.6.) Let C_j be the subarc joining q_0 to q_j of C_0, then

$$l_x^*(C_j) \leq l_x^*(C_0). \tag{2.15}$$

Since ρ_j is a g^*-shortest arc in D, and $l \cup C_j \cup \varphi_j$ is piecewise C^1.
By Lemma 2.9,

$$l_x^*(\rho_j) \leq l_x^*(l) + l_x^*(C_j) + l_x^*(\varphi_j).$$

By Corollary 2.3 and (2.15),

$$l_x^*(\rho_j) \leq l_x^*(l) + l_x^*(C_0) + 2\alpha r. \tag{2.16}$$

Clearly, the right side of (2.16) is independent on j, so let $j \to \infty$, then $l_x^*(\rho) < \infty$.
In Case 2, let Y be the portion of the y-axis within D. Then either,

Case 2.1: There is a sequence $t_j \to \infty$ such that $d_D^*(\rho(t_j), Y) \leq 1$ for all j; or

Case 2.2: There is $\delta_1 > 0$, such that $d_D^*(\rho(t), Y) > 1$ for any $t > \delta_1$.

If Case 2.1 is true, then one can assume that $d_D^*(\rho(t_j), l) > 2$ for every j, because $\rho|_{[0,s]}$ is a g^*-shortest arc in D for any $s > 0$, $l^*(\rho) = \infty$, and l is compact in D. Since $d_D^*(\rho(t_j), Y) \leq 1$, there is a piecewise C^1 arc r_j in D, which is free of self-intersections and joins $p_j = \rho(t_j)$ to a point $p_j' \in Y$ with

$$l_x^*(r_j) \leq l^*(r_j) < 2. \tag{2.17}$$

Since $p_j' \in Y$, $x(p_j') = 0$, by Lemma 2.10, there is x-chain φ_j, which starts at p_j' and ends at $q_j \in C_0$, with ρ_j and C_j as defined in case 1, (see Figure 2.7.) then

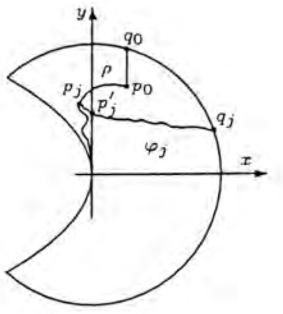

Figure 2.7: Case 2.1.

$$\begin{aligned} l_x^*(\rho_j) &= l_x^*(l_j) + l_x^*(\varphi_j) + l_x^*(C_j) + l_x^*(l) \\ &= 2 + 2\alpha r + l_x^*(C_0) + l_x^*(l). \end{aligned}$$

Let $j \to \infty$, then $l_x^*(\rho) < \infty$.

For Case 2.2.

(a) $\rho \cap Y = \emptyset$. Let l be the horizontal arc in $\mathbb{R}^2$ joining p_0 to a point $q_0 \in Y$, and let $\rho(\delta_2)$ be the last point of ρ which lies on l. Let K denote the region bounded by l and the portion of ρ from $\rho(\delta_2)$ onward and the portion of y-axis from $(0,0)$ to q_0. (See Figure 2.8.) $l^*(\rho) = \infty$, so there exists $t_0 > \delta_1$, such that

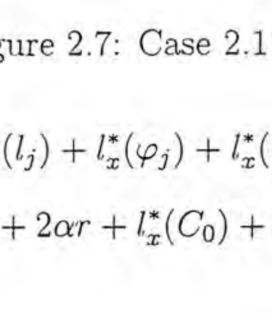

Figure 2.8: K shaded; $\rho \cap Y = \emptyset$.

$d_D^*(\rho(t), l) > 1$ if $t > t_0$. Set $t_j = t_0 + 2j, j \in \mathbb{N}$, $p_j = \rho(t_j)$ and

$$D_j = \{p \in D \mid d_D^*(p, p_j) < 1\}.$$

Then $D_j \cap D_k = \emptyset$ if $j \neq k$, and $D_j \cap \{p \in D \mid x \text{ coordinate of } p \geq 0\} = \emptyset$.

For any fixed p_j, let L be the tangent line to $F(\rho)$ at $F(p_j)$. F is a C^1 immersion, so one can find a neighborhood $U_j \subset D_j$ of p_j, such that F is injective on U_j. Then L cuts the tangent plane at $F(p_j)$ into two parts consisting of vectors pointing " into " and " out of " $F(K \cap U_j)$. If $p_j \in \partial D$, by Lemma 2.8, one obtains that there exists a closed half disc $\triangle$ in $\mathbb{R}^2$, such that $\triangle \cap L$ is the diameter and $\triangle \subset F(U_j)$, moreover, $F(\rho \cap U_j) \cap Int(\triangle) = \emptyset$. If $p_j \notin \partial D$, then some portion of $F \circ \rho$ through $F(p_j)$ is a subarc of L, of course, such a $\triangle$ exists.

Given any vector at $F(p_j)$ pointing " into " $F(K \cap U_j)$, let β be the line segment in $\triangle$ starting at $F(p_j)$ tangent to the given vector and parametrized by the ordinary Euclidean arclength. F is an isometric immersion on U_j, so β can be pulled back to a g^*-shortest arc β_* in K. Moreover β_* can be extended to a g^*-shortest arc in K with g^*-length equal to 1. Otherwise, β_* would again meet ∂K at some point q_j with the g^*-length < 1, note that $d_D^*(\rho(t_j), Y) > 1$, and that $d_D^*(\rho(t_j), l) > 1$, then β_* could only meet ρ again, so $\beta_* \subset \rho$ and the vector tangent to $F \circ \rho$ is also the vector tangent to $F \circ \beta_*$. It contradicts with the choice of the given vector. Hence the g^*-area of $D_j \cap K$ is no less than $\pi/2$.

From above, one obtains

$$\text{area}^*(K) \geq \sum_{j=1}^{\infty} \text{area}^*(D_j \cap K) = \infty.$$

From $-\alpha < \lambda_1 < \lambda_2 < \alpha$, one obtains

$$|\text{Jacobian } F| \leq \alpha^2.$$

So $\text{area}^*(K) \leq \alpha^2 \text{area}(K)$. Since D is bounded in $\mathbb{R}^2$, then $\text{area}(K) < \infty$. Hence

$$\infty \leq \text{area}^*(K) \leq \alpha^2 \text{area}(K) < \infty.$$

It is impossible.

(b) $\rho \cap Y \neq \emptyset$. Let $\rho(\delta_2)$ be the last point of ρ which lies on the y-axis, let K denote the region bounded by the portion of ρ from $\rho(\delta_2)$ onward, and the portion of y-axis from $(0,0)$ to $\rho(\delta_2)$. (See Figure 2.9.) Let $t_0 = \delta_1 + 1$, the following

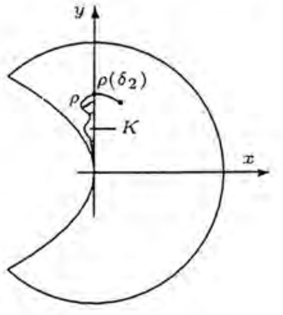

Figure 2.9: K shaded; $\rho \cap Y \neq \emptyset$.

procedure is the same as (a).

Therefore $l_x^*(\rho) < \infty$.

Since

$$\infty = l^*(\rho) \leq l_x^*(\rho) + l_y^*(\rho)$$

and $l_x^*(\rho) < \infty$, then $l_y^*(\rho) = \infty$. $\square$

Definition 2.11 *A parametrized arc h in $\mathcal{D}$ is prehorizontal if $F \circ h$ is a horizontal line segment in $\mathbb{R}^2$, and is parametrized by its ordinary arclength. (It may be open, half open, or closed, finite or infinite.)*

Let $\rho(t)$, $0 \leq t < \infty$, be a distinguished ray, where $t = g^*$-arclength parameter, define $u = v(t) = l_y(F \circ \rho|_{[0,t]})$, for $t \geq 0$. Since F and ρ are C^1, then v is C^1 and

$$0 \leq \frac{dv}{dt} \leq 1$$

for all $t \in [0, \infty)$. Moreover, by Lemma 2.12, $v(t) \to \infty$ as $t \to \infty$. For any number $u = v(t)$, let h_u denote a maximal prehorizontal arc in D through $\rho(t)$. By the definition of u, if $u = v(t_1) = v(t_2)$ and $t_1 \neq t_2$, then the subarc joining $\rho(t_1)$ to $\rho(t_2)$ on ρ is mapped to a horizontal arc. So h_u is uniquely determined for any $u \in [0, \infty)$ up to changes in its g^*-arclength parametrization.

Corollary 2.4 *For any fixed distinguished ray ρ, the arcs $h_u \to$ the origin as $u \to \infty$.*

Proof Fix a distinguished ray ρ, and let h_u be the prehorizontal arc through $\rho(t)$, where $u = v(t) = l_y(F \circ \rho|_{[0,t]})$.

For $k = 1, 2, \ldots$, let

$$D(k) = \{(x, y) \in D \mid x^2 + y^2 \geq r^2/2^k\},$$

where r is the radius of C. (See Figure 2.9.) Assume that $\rho(0) \in D(k)$ for some $k \in \mathbb{N}$. $\rho(t) \to 0$ as $t \to \infty$, so ρ will leave $D(k)$. Let $\rho(t_0)$ be the last point of ρ in $D(k)$. If $h_u \cap D(k) \neq \emptyset$ where $u = v(t)$ and $t > t_0$, then h_u must intersect the circular arc

$$C(k) = \{(x, y) \in D \mid x^2 + y^2 = r^2/2^k\}.$$

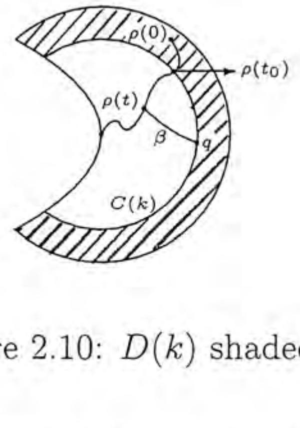

Figure 2.10: $D(k)$ shaded.

Let β be a subarc of h_u joining $\rho(t)$ to a point $q \in C(k)$, then

$$t - t_0 \leq l^*(\beta) + l^*(C(k)),$$

since ρ is a distinguished ray and parametrized by its g^*-arclength.

By Lemma 2.9,

$$l^*_x(\beta) \leq l^*_x(\rho) + l^*_x(C(k)).$$

β is a subarc of a prehorizontal arc h_u, so $l^*_x(\beta) = l^*(\beta)$.

Hence

$$\begin{aligned} t - t_0 &\leq l^*_x(\beta) + l^*(C(k)) \\ &\leq l^*_x(\rho) + l^*_x(C(k)) + l^*(C(k)). \end{aligned} \tag{2.18}$$

By Lemma 2.12, the right hand side of (2.18) is finite, so the inequality (2.18) can't hold for all sufficient large t. Note that u is increasing, then the inequality (2.18) can't hold for all sufficient large u.

Hence the corollary is true. □

Let $P_0 = (P \cap D) \setminus C$, where P is the parabola $cy^2 = -x$ in $\mathbb{R}^2$, and let P_0^+ and P_0^- be the components of P_0 lying above and below the x-axis, respectively, (See Figure 2.11.) then

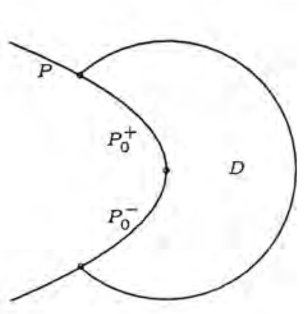

Figure 2.11: $P_0 = (P \cap D) \setminus C$.

Lemma 2.13 *Assume that D is complete with respect to the d_D^* distance. Fix a distinguished ray ρ, let Ω be the subset of $\mathbb{R}^+$ consisting of all u for which the prehorizontal section h_u*

(A) intersects ρ transversally, and

(B) intersects ∂D transversally, with one endpoint $b^+(u)$ on P_0^+ and the other $b^-(u)$ on P_0^-. (See Figure 2.12.)

Then, Ω is open and $m(\mathbb{R}^+ \setminus \Omega) < \infty$, where m denotes the Lebesgue measure on $\mathbb{R}$.

Proof If $u \in \Omega$, then $h_u \cap \rho = \{$only one point in $D\}$. Otherwise, $h_u \cap \rho$ would include at least two distinct points $\rho(t_1)$ and $\rho(t_2)$, since h_u and ρ are two g^*-shortest arcs in D, and h_u is prehorizontal, then h_u and ρ would be same graph between $\rho(t_1)$ and $\rho(t_2)$. It is contradiction with the assumption (A). Similarly, if $u \in \Omega$, then h_u intersects ∂D only one point $b^+(u) \in P_0^+$ and only one point

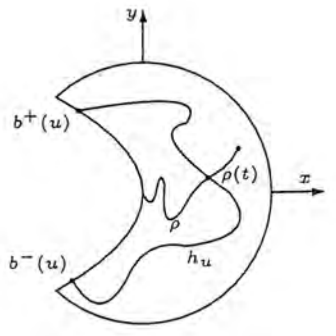

Figure 2.12: $\rho \cap h_u = \rho(t)$.

$b^-(u) \in P_0^-$. Otherwise, there exists the third point $p \in h_u \cap \partial D$, so one of the three points $b^+(u), b^-(u)$ and p must be an interior point of h_u, by Remark 2.2, h_u must be tangent to ∂D at the interior of h_u, it is contradiction with the assumption (B).

To show that Ω is an open subset of $\mathbb{R}^+$. Fix any $u \in \Omega$, there exists a neighborhood N of h_u in D, such that $F|_N$ is injective. Since $F|_{h_u}$ is injective and h_u is compact. By above discuss, one has known that $h_u \cap \partial D = \{b^+(u), b^-(u)\}$ and that $h_u \cap C = \emptyset$, noting that h_u intersects ∂D and ρ transversally, so one can choose N smaller if necessary, such that $N \cap \partial D = N \cap P_0$ and $F(N \cap P_0)$ and $F(N \cap \rho)$ nowhere have a horizontal tangent. One obtains that $F(h_u)$ is a horizontal line segment which intersects $F(N \cap P_0)$ and $F(N \cap \rho)$ transversally, with one endpoint on $F(N \cap P_0^+)$, and the other on $F(N \cap P_0^-)$ (see Figure 2.13). If u' is close to u, then $F(h_{u'})$ is a horizontal segment which intersects $F(N \cap$

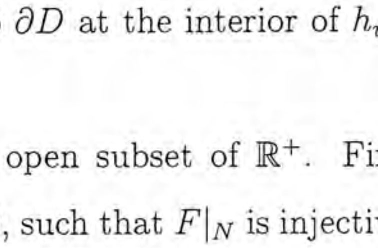

Figure 2.13: $F(N \cap D)$ shaded in $F(N)$.

$P_0)$, $F(N \cap \rho)$ transversally, and intersects $F(N \cap P_0)$ one point in $F(N \cap P_0^+)$ and one point in $F(N \cap P_0^-)$. So Ω is open.

By Corollary 2.4, $h_u \to 0$ as $u \to \infty$, so there exists $u_0 \in \mathbb{R}^+$, such that $d(h_u, 0) < d(\rho(0), 0)$ for all $u > u_0$. One wants to prove that $m((\mathbb{R}^+ \setminus \Omega) \cap (u_0, \infty)) = 0$. Let $y^*(p)$ be the y coordinate at $F(p)$ for $p \in \mathcal{D}$, and let W_1 be the set of critical values of the C^1 function

$$u = v(t) = \int_0^t |\frac{dy^*(\rho(s))}{ds}| ds.$$

Then W_1 is closed and $m(W_1) = 0$. Define a function $Y^* : \mathbb{R}^+ \setminus W_1 \longrightarrow \mathbb{R}$ by

$$Y^*(u) = y^*(\rho(v^{-1}(u))),$$

then the function is C^1 on $\mathbb{R}^+ \setminus W_1$, and $dY^*/du = \pm 1$, because

$$\frac{dY^*}{du} = \frac{dy^*}{dt} \cdot \frac{dt}{du} = \pm \frac{dv}{dt} \cdot \frac{dt}{dv} = \pm 1.$$

Let V be the set of critical values of the function y^* restrict to P_0, and let $W_2 = Y^{*-1}(V)$. By Sard's Theorem, $m(V) = 0$. Since Y^* is a C^1 immersion on $\mathbb{R}^+ \setminus W_1$, $m(W_2) = 0$.

One wants to claim that

$$((\mathbb{R}^+ \setminus (W_1 \cup W_2)) \cap (u_0, \infty)) \subset (\Omega \cap (u_0, \infty)).$$

If $u \in (\mathbb{R}^+ \setminus (W_1 \cup W_2))$, then $F \circ h_u$ is a closed finite line segment. Since, if $F \circ h_u$ is infinite, then one can obtain a prehorizontal arc $h'_u : [0, \infty) \longrightarrow D$ along h_u, and then h'_u is a distinguished ray in D. By Lemma 2.12, $l^*_x(h'_u) < \infty$, it is contradiction with that $l^*_x(h'_u) = l^*(h'_u) = \infty$, so $F \circ h_u$ is finite. If $F \circ h_u$ is a finite line segment excluding either one of its endpoints, say p, then $\{F^{-1}(p_n)\}$ in D is a Cauchy sequence with respect to d^*_D distance, where $\{p_n\}$ is a sequence of points in $F \circ h_u$ which converges to p, noting that D is complete, so the sequence $\{F^{-1}(p_n)\}$ converges to a point $q \in D$, hence h_u can be extended to a prehorizontal arc including q, it is contradiction with that h_u is maximal. So the endpoints of h_u exist and must be on P_0 since $h_u \cap C = \emptyset$. $u \notin W_1$, so h_u intersects ρ transversally. $u \notin W_2$, so h_u intersects P_0 transversally. Suppose that the endpoints of h_u both lie on P_0^+, or both lie on P_0^-, then h_u and a closed arc of P_0 bound a compact set $R \subset D$, ρ intersects h_u transversally at $\rho(v^{-1}(u))$, so it can't intersect h_u again, it is contradiction with that $\rho(t) \to 0$ as $t \to \infty$. Hence the endpoints of h_u lie on opposite sides of the x-axis on P_0. Thus $u \in \Omega$. Therefore

$$((\mathbb{R}^+ \setminus \Omega) \cap (u_0, \infty)) \subset ((W_1 \cup W_2) \cap (u_0, \infty)).$$

Clearly,

$$m((\mathbb{R}^+ \setminus \Omega) \cap (u_0, \infty)) \leq m((W_1 \cup W_2) \cap (u_0, \infty)) = 0.$$

Therefore

$$m(\mathbb{R}^+ \setminus \Omega) < \infty. \quad \square$$

Definition 2.12 *Given a point q on the positive x-axis, there exist two lines L^+ and L^- through p in $\mathbb{R}^2$ tangent to the parabola P from above and below respectively. Let $G(q)$ denote the finite, closed region bounded by an arc of P and the segments of L^+ and L^- (see Figure 2.14).*

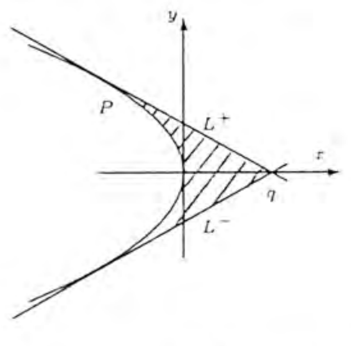

Figure 2.14: $G(q)$ shaded.

Definition 2.13 *Given $u \in \Omega$, there exists a left-most point $q(u)$ on the positive x-axis, such that $h_u \subset G(q(u))$. Let $G(u)$ denote $G(q(u))$, and let $A(u)$ denote the Euclidean area of $G(u)$. The lines along the boundary $G(u)$ are denoted by $L^+(u)$ and $L^-(u)$, respectively. (See Figure 2.15.)*

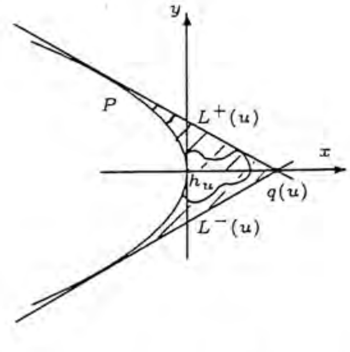

Figure 2.15: $G(u)$ shaded.

Corollary 2.5 *Under the hypothesis of Lemma 2.13, the function $A(u)$ is strictly decreasing on Ω and $A(u) \to 0$ as $u \to \infty$ in Ω.*

Proof By Corollary 2.4, $h_u \to 0$ as $u \to \infty$, so $A(u) \to 0$ as $u \to \infty$ in Ω. Let $u_1, u_2 \in \Omega$ and $u_1 > u_2$, then $h_{u_1} \cap h_{u_2} = \emptyset$. Let K be the domain bounded by the arc P and h_{u_2}, and K includes the origin. (See Figure 2.16.) Since $u_2 \in \Omega$, then

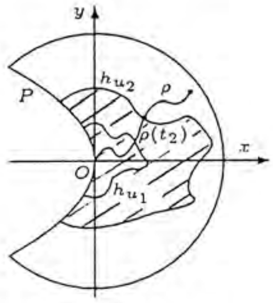

Figure 2.16: K shaded and $o \in K$.

h_{u_2} intersects ρ transversally at only one point, say $\rho(t_2)$, hence, $\rho|_{[t_2,\infty)} \subset K$. Let h_{u_1} intersect ρ at $\rho(t_1)$, then $t_1 > t_2$, so $\rho(t_1) \in K$ and $h_{u_1} \subset K$. Thus $A(u)$ is strictly decreasing. $\square$

Let $g(u) = l^*(h_u) > 0$ for $u \in \Omega$, then $g(u)$ is continuous on Ω since $b^+(u)$ and $b^-(u)$ are continuous on Ω. For any fixed $u \in \Omega$, let

$$H(u) = \bigcup_{v \in \Omega, v > u} h_v,$$

and for any open interval $I \subset (\Omega \cap (u, \infty))$, let

$$H_I = \bigcup_{v \in I} h_v.$$

Using that $dY^*/du = \pm 1$ and is continuous on Ω, one obtains that $dY^*/du = 1$ or -1 throughout I, so Y^* is injective on $I \subset (\Omega \cap (u, \infty))$. Hence F is a C^1 imbedding on H_I. Thus

$$\text{area}^* H_I = \int_I g(v) dv.$$

$\Omega \cap (u, \infty)$ is the union of disjoint open intervals I, then let

$$V^*(u) = \text{area}^* H(u) = \sum_I area^* H_I.$$

$|\text{Jacobian } F| \leq \alpha^2$ on $\mathcal{D}$, so

$$V^*(u) \leq \alpha^2 \text{area } H(u) \leq \alpha^2 \cdot \text{ area of } D \leq \alpha^2 \pi r^2 < \infty.$$

Lemma 2.14 *The function $V^* : \Omega \longrightarrow \mathbb{R}^+$ is strictly decreasing, with*

$$\frac{dV^*}{du} = -g(u).$$

Proof Since Ω is open, the lemma follows from the definition of $V^*(u)$. □

Lemma 2.15 *Assume that D is complete with respect to the d_D^* distance. Let ρ be a distinguished ray, and let $\Omega \subset \mathbb{R}^+$ be the open set defined in Lemma 2.13. Then there exists an open subset $W \subset \Omega$, such that the Lebesgue measure $m(\Omega \setminus W)$ of $\Omega \setminus W$ is finite, and*

$$(u_1 - u) A(u_1)^{1/3} \leq b A(u)^{2/3} \qquad \forall u, u_1 \in W, \tag{2.19}$$

where $b > 0$ is a constant .

Proof For any $u \in \Omega$, set $V(u) = \text{area } H(u)$, then

$$V^*(u) \leq \alpha^2 V(u) \leq \alpha^2 A(u),$$

where $A(u)$ is defined in Definition 2.14. Let $X(u) = x$ coordinate at $q(u)$ (defined in Definition 2.14), then $X(u)$ is strictly decreasing on Ω. Let $X_0 = \sup\{X(u); u \in \Omega\}$. Given $u_1 > u$ in Ω, let $h = h_u$, $h_1 = h_{u_1}$, $L^+ = L^+(u)$, $L^- = L^-(u)$, $L_1^+ = L^+(u_1)$, $L_1^- = L^-(u_1)$, $G = G(u)$, $G_1 = G(u_1)$, $A = A(u)$, $A_1 = A(u_1)$, $V = V(u)$, $V_1 = V(u_1)$, $V^* = V^*(u)$, $V_1^* = V^*(u_1)$, $g = g(u)$, $g_1 = g(u_1)$, $X = X(u)$, $X_1 = X(u_1)$. Clearly, $X_1 < X$, $G_1 \subset G$, and $u_1 - u = l_y^*(\gamma)$, where $\gamma = \rho|_{[\rho(v^{-1}(u)),\rho(v^{-1}(u_1))]}$. By the definition of G_1, one obtains $h_{u_1} \cap (L_1^+ \cup L_1^-) \neq \emptyset$.

There exists $p_1 \in h_{u_1} \cap L_1^+$ with $p_1 \notin L_1^-$. L_1^+ is straight line, so $\xi_1 = x\cos\theta_1 + y\sin\theta_1$ is a constant $\xi_1' > 0$ on L_1^+, with that θ_1 is the positive acute angle which is formed by L_1^+ and y-axis. (See Figure 2.17.) The endpoints of h_1

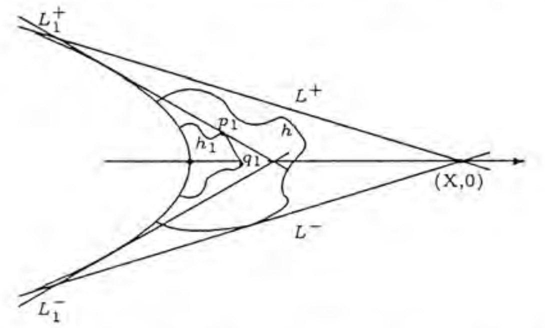

Figure 2.17: q_1 is any right-most point on h_1.

lie on opposite sides of the x-axis, so h_1 must cross the positive x-axis, hence if q_1 is any right-most point on h_1, the x coordinate at q_1 must be positive. By Lemma 2.10, there is an x-chain φ joining q_1 to ∂D, in particular to $C_0 = C \cap \partial D$, then $\varphi \cap h \neq \emptyset$ as x is increasing along φ. Let φ_1 be the arc of φ from q_1 to its first intersection q_0 with h. (See Figure 2.18.)

By Lemma 2.11,

$$l_x^*(\varphi_1) \leq \alpha l_x(\varphi_1) \leq \alpha X.$$

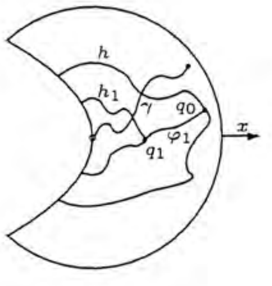

Figure 2.18: x-chain φ_1.

By Lemma 2.9,

$$l_x^*(\gamma) \leq l_x^*(h_1) + l_x^*(\varphi_1) + l_x^*(h).$$

Thus

$$l_x^*(\gamma) \leq g_1 + g + \alpha X. \tag{2.20}$$

Since P is the parabola $x = -cy^2$, one has

$$X = (\frac{3}{2}\sqrt{c}A)^{2/3}, X_1 = (\frac{3}{2}\sqrt{c}A_1)^{2/3}. \tag{2.21}$$

So (2.20) can be rewritten by

$$l_x^*(\gamma) \leq g_1 + g + \alpha_0 A^{2/3}, \tag{2.22}$$

where $\alpha_0 = \alpha(\frac{3}{2}\sqrt{c})^{2/3} > 0$.

Using $y \sin\theta_1 = \xi_1 - x\cos\theta_1$, and the triangle inequality, one obtains

$$l_y^*(\gamma)\sin\theta_1 \leq l_{\xi_1}^*(\gamma) + l_x^*(\gamma)\cos\theta_1.$$

On the other hand,

$$\begin{aligned}
\sin\theta_1 &= \frac{2X_1}{\sqrt{\frac{X_1}{c} + 4X_1^2}} \\
&= 2(\frac{X_1 c}{4X_1 c + 1})^{1/2} \\
&> 2\frac{(cX_1)^{1/2}}{\sqrt{4cX_0 + 1}} \\
&= \alpha_1 A_1^{1/3},
\end{aligned}$$

where $\alpha_1 = (12c^2)^{1/3}/\sqrt{4cX_0+1}$.

So

$$\begin{aligned} l^*_{\xi_1}(\gamma) &\geq l^*_y(\gamma)\sin\theta_1 - l^*_x(\gamma) \\ &\geq \alpha_1 l^*_y(\gamma)A_1^{1/3} - l^*_x(\gamma). \end{aligned} \tag{2.23}$$

By (2.22) and (2.23),

$$l^*_{\xi_1}(\gamma) \geq \alpha_1 l^*_y(\gamma)A_1^{1/3} - (g + g_1 + \alpha_0 A^{2/3}). \tag{2.24}$$

On the other hand, ξ_1 is a constant $\xi_1' > 0$ along L_1^+, so $\xi_1 \leq \xi_1'$ on G and on the parabola P. By Lemma 2.10, there exists a ξ_1-chain β from p_1 to a point of $C_0 = C \cap D$, and β must cross h to reach C_0. Let β' be the arc of β from p_1 to its first intersection $p_0 \in h$. (See Figure 2.19.)

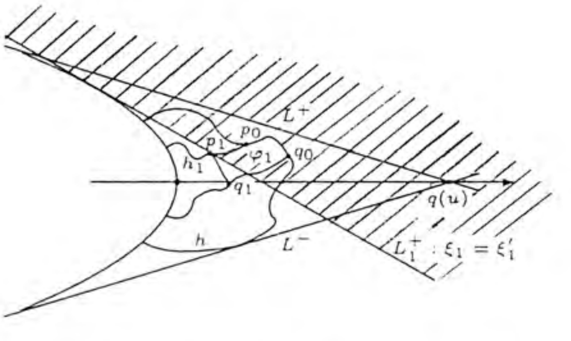

Figure 2.19: The region $\xi_1 \geq \xi_1'$ shaded.

By Lemma 2.9,

$$l^*_{\xi_1}(\gamma) \leq l^*_{\xi_1}(h) + l^*_{\xi_1}(\beta') + l^*_{\xi_1}(h_1).$$

By Lemma 2.11,

$$l^*_{\xi_1}(\gamma) \leq g + g_1 + \alpha l_{\xi_1}(\beta').$$

Since $l_{\xi_1}(\beta') = d(p_0, L_1^+) \leq d(q(u), L_1^+) < X$, then

$$\begin{aligned} l^*_{\xi_1}(\gamma) &\leq g + g_1 + \alpha X \\ \text{by (2.21)} \qquad &= g + g_1 + \alpha_0 A^{2/3}. \end{aligned} \tag{2.25}$$

By (2.24) and (2.25) and $u_1 - u = l_y^*(\gamma)$,

$$\alpha_1 l_y^*(\gamma) A_1^{1/3} - (g + g_1 + \alpha_0 A^{2/3}) \le l_{\xi_1}^*(\gamma) \le g + g_1 + \alpha_0 A^{2/3},$$

so

$$\alpha_1 (u_1 - u) A_1^{1/3} \le 2(g + g_1 + \alpha_0 A^{2/3}),$$

hence

$$(u_1 - u) A_1^{1/3} \le \alpha_2 (g + g_1) + \alpha_3 A^{2/3},$$

where $\alpha_2 = 2/\alpha_1$, $\alpha_3 = 2\alpha_0/\alpha_1$ are two constants.

By Lemma 2.14,

$$(u_1 - u) A_1^{1/3} \le \alpha_2 \left(\left| \frac{dV^*}{dv}(u) \right| + \left| \frac{dV^*}{dv}(u_1) \right| \right) + \alpha_3 A^{2/3}. \tag{2.26}$$

Define $f_1(v) = A^{1/3}(v)$, $f_2(v) = V^{*1/3}(v)$, $v \in \Omega$, then (2.26) can be rewritten by

$$(u_1 - u) f_1(u_1) \;\; \le \;\; \alpha_2 \left(\left| 3 f_2^2(u) \frac{df_2}{dv}(u) \right| + \left| 3 f_2^2(u_1) \frac{df_2}{dv}(u_1) \right| \right) + \alpha_3 f_1^2(u).$$

$V^*(u_1) \le V^*(u) \le \alpha^2 A(u)$, so $f_2(u_1) \le f_2(u) \le \alpha^{2/3} f_1(u)$. Hence

$$(u_1 - u) f_1(u_1) \le f_1^2(u) \left[3\alpha_2 \alpha^{4/3} \left(\left| \frac{df_2}{dv}(u) \right| + \left| \frac{df_2}{dv}(u_1) \right| \right) + \alpha_3 \right]. \tag{2.27}$$

Similarly, if $q_1 \in L_1^-$ but $q_1 \notin L_1^+$, then taking $\xi_1 = x\cos(-\theta_1) + y\sin(-\theta_1)$, (2.27) is valid also.

By Lemma 2.14, one knows that $f_2(v) > 0$ and is strictly decreasing, moreover df_2/dv is continuous on Ω. Let

$$\Omega_1 = \{ u' \in \Omega \mid \frac{df_2}{dv}(u') \le -1 \},$$

then Ω_1 is closed and $m(\Omega_1) < \infty$, since f_2 is bounded and always positive.

Let $W = \Omega \setminus \Omega_1$, and let $b = 6\alpha_2\alpha^{4/3} + \alpha_3$, then

$$(u_1 - u) f_1(u_1) \le b f_1^2(u) \qquad \forall u_1, u \in W,$$

i.e.,

$$(u_1 - u)A^{1/3}(u_1) \leq bA^{2/3}(u).$$

Hence, the lemma holds. □

Proof of the Main Lemma Use Lemma 2.5 and Lemma 2.15, it is easy to know that the Main Lemma is true. □

2.3 Proof of Lemma 2.3

Lemma 2.3 *If M is a complete oriented surface C^2 immersed in $\mathbb{R}^3$ with $K \leq -\kappa < 0$, then $M_{I\!I\!I}$ is pseudo convex.*

Proof Suppose that $M_{I\!I\!I}$ is not pseudo convex. Then $M_{I\!I\!I}$ is cancave at some point $p \in \partial\widetilde{M}_{I\!I\!I}$, i.e., there exists an open set $U \subset M_{I\!I\!I}$, such that $p \in \widetilde{U} \subset \widetilde{M}_{I\!I\!I}$ and the extended Gauss map $\widetilde{N}$ is injective on $\widetilde{U}$, moreover $N(U)$ contains the interior of an exterior rectangle $R(\gamma, \sigma)$ through $\widetilde{N}(p)$. Rotate M in $\mathbb{R}^3$, such that $\widetilde{N}(p) = (0, 0, 1)$ and that γ is in the hemisphere $y \geq 0$ on $\mathbb{S}^2$. Replace $R(\gamma, \sigma)$ and U by smaller ones, such that $z > \frac{\sqrt{2}}{2}$ for any point $(x, y, z) \in R(\gamma, \sigma)$, and that $N(U)$ is exactly the interior of the exterior rectangle $R(\gamma, \sigma)$.

Let μ be a function on the open Northern hemisphere $\mathbb{S}^2_+$ of $\mathbb{S}^2$, defined by

$$\mu(x, y, z) = (-y/z, x/z) \qquad \text{for } (x, y, z) \in \mathbb{S}^2_+.$$

It is easy to see that μ is injective on $\mathbb{S}^2_+$, and that the inverse function μ^{-1} is given by

$$\mu^{-1}(x, y) = \left(\frac{y}{\sqrt{1 + x^2 + y^2}}, \frac{-x}{\sqrt{1 + x^2 + y^2}}, \frac{1}{\sqrt{1 + x^2 + y^2}}\right) \quad \forall (x, y) \in \mu(\mathbb{S}^2_+).$$

Clearly, $\mu(N(p)) = (0, 0)$, $\mu(\gamma)$ is in the half plane $x \leq 0$ and is an arc of some non-linear conic section through $(0, 0)$. Let π be the vertical projection on U, i.e.,

$$\pi(x, y, z) = (x, y) \qquad \forall (x, y, z) \in U.$$

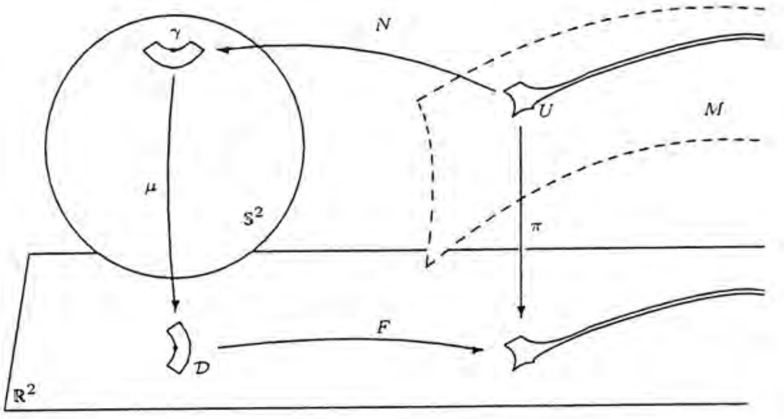

Figure 2.20: Mapping F.

Clearly, π is a C^2 immersion. Let $\mathcal{D} = \mu(N(U))$, and let $F = \pi \circ N^{-1} \circ \mu^{-1}$, then $F : \mathcal{D} \longrightarrow \mathbb{R}^2$ is a C^1 immersion.

To show that the eigenvalues of the differential dF are real, of opposite sign, and uniformly bounded on $\mathcal{D}$. for any $q \in U$, there is a sufficiently small neighborhood of q on U, such that it can be given by

$$r(u,v) = (x(u,v), y(u,v), z(u,v)),$$

where (u,v) is in some neighborhood in $\mathbb{R}^2$, and there exists a local inverse function π^{-1} for π. Then

$$N(x,y,z) = \frac{r_u \times r_v}{|r_u \times r_v|} = (\frac{y_u z_v - y_v z_u}{|r_u \times r_v|}, \frac{-(x_u z_v - x_v z_u)}{|r_u \times r_v|}, \frac{x_u y_v - x_v y_u}{|r_u \times r_v|}),$$

and $x_u y_v - x_v y_u > 0$ since

$$\frac{x_u y_v - x_v y_u}{|r_u \times r_v|} > \frac{\sqrt{2}}{2}.$$

By the inverse function theorem, u, v can be written by $u = u(x,y)$ and $v = v(x,y)$ respectively in some neighborhood of q on U.

Hence any small enough neighborhood V of q on U can be given by

$$r(x,y) = (x, y, f(x,y))$$

for some C^2 function f.

Given any point $(x, y, f(x, y)) \in V$,

$$N(x, y, f(x, y)) = (-f_x, -f_y, 1)/(1 + f_x^2 + f_y^2)^{1/2}$$

and

$$\frac{1}{(1 + f_x^2 + f_y^2)^{1/2}} > \frac{\sqrt{2}}{2}.$$

So $f_x^2 + f_y^2 < 1$ and $\mu \circ N(x, y, f(x, y)) = (f_y, -f_x)$. Hence any local inverse $F^{-1} = \mu \circ N \circ \pi^{-1}$ for F has the form

$$F^{-1}(x, y) = (f_y, -f_x).$$

Thus

$$dF^{-1} = \begin{pmatrix} f_{yx} & f_{yy} \\ -f_{xx} & -f_{xy} \end{pmatrix}$$

The Gauss curvature $K \leq -\kappa < 0$ on M, so

$$\begin{aligned} \begin{vmatrix} f_{yx} & f_{yy} \\ -f_{xx} & -f_{xy} \end{vmatrix} &= f_{xx}f_{yy} - f_{xy}^2 \\ &= \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}(1 + f_x^2 + f_y^2)^2 \\ &= K(1 + f_x^2 + f_y^2)^2 \leq -\kappa < 0. \end{aligned}$$

So the eigenvalues λ_1 and λ_2 of the differential dF^{-1} satisfy

$$\begin{cases} \lambda_1 + \lambda_2 = 0, \\ \lambda_1\lambda_2 \leq -\kappa < 0. \end{cases}$$

Hence λ_1 and λ_2 are real , of opposite sign, and satisfy

$$|\lambda_1| = |\lambda_2| \geq \sqrt{\kappa}.$$

Clearly, κ is independent of the choice of F^{-1} for F. Thus the eigenvalues of the differential dF are real, of opposite sign, and uniformly bounded by $\frac{1}{\sqrt{\kappa}}$ on $\mathcal{D}$. Choose positive constants c and r, such that

$$D = \{(x, y) \mid 0 < x^2 + y^2 \leq r^2; cy^2 \geq -x\} \subset \mathcal{D}.$$

Clearly, D is closed in $\mathcal{D}$. Let g^* be the Riemannian metric on $\mathcal{D}$ induced by F. Define the distance $d_D^*(p_1, p_2)$ between points p_1 and p_2 on D to be infimum of the g^*-lengths of paths in D from p_1 to p_2.

To show that D with respect to the metric d_D^* is a complete metric space. Let $\bar{g}^*$ be the Riemannian metric on U induced by $\mu \circ N$. Clearly, $\bar{g}^*$ is exactly the metric on U induced from the Euclidean metric by π. Let $W = N^{-1} \circ \mu^{-1}(D)$, then $\bar{g}^*$ is the metric on W. Moreover W is closed in M. Otherwise, there exists a limit point q of W in $M \backslash W$. The Gauss map $N : M \longrightarrow \mathbb{S}^2$ is continuous, so $N(q)$ is in the closure of $N(W)$ in $\mathbb{S}^2$. On the other hand, $\widetilde{N} : \widetilde{W} \longrightarrow \mathbb{S}^2$ is injective and $q \notin W$, so $N(q) \notin N(W)$. Note that the only limit point of $N(W) = \mu^{-1}(D)$ which is in $\mathbb{S}^2 \setminus N(W)$ is the North pole $\widetilde{N}(p)$. Hence $\widetilde{N}(p) = N(q) = \widetilde{N}(q)$. Thus $p = q$, it is contradiction with that $p \in \partial\widetilde{M}_{I\!I}$ and $q \in M \setminus W$. Define the distance $d_W^*(q_1, q_2)$ between points q_1 and q_2 on W as the infimum of the $\bar{g}^*$-lengths of paths in W from q_1 to q_2.

W is isometric to D, so it is sufficient to show that W with the metric d_W^* is a complete metric space. Of course, M is complete and W is closed, so W is complete. Define the distance $d_W(q, q')$ between points q and q' on W as the infimum of the ordinary I-lengths of arcs in W joining q to q', then W is still complete with the metric d_W since W is a submanifold of M with piecewise smooth boundary. (Cf [[7], Appendix 3].) Hence, sufficient to show that

$$d_W^*(q_1, q_2) \leq d_W(q_1, q_2) \leq \sqrt{2} d_W^*(q_1, q_2) \quad \forall q_1, q_2 \in W \tag{2.28}$$

Since, $d_W(q_1, q_2) \leq \sqrt{2} d_W^*(q_1, q_2)$ implies that any d_W^* Cauchy sequence is a d_W Cauchy sequence, so it converges in the d_W metric. On the other hand, $d_W^*(q_1, q_2) \leq d_W(q_1, q_2)$ implies that convergence in the d_W metric is convergence in the d_W^* metric.

To check (2.28). Given any arc γ in W by

$$\gamma(t) = (x(t), y(t), f(x(t), y(t))).$$

where t is a $\bar{g}^*$-length parameter for γ, the ordinary arclength $s(t)$ is given by

$$s(t) = \int_0^t ((x'(\tau))^2 + (y'(\tau))^2 + (f_x x'(\tau) + f_y y'(\tau))^2)^{1/2} d\tau.$$

$(x')^2 + (y')^2 = 1$ and $f_x^2 + f_y^2 < 1$, so

$$\begin{aligned} s(t) &= \int_0^t (1 + (f_x x'(\tau) + f_y y'(\tau))^2)^{1/2} d\tau \\ &\leq \int_0^t (1 + (f_x^2 + f_y^2)((x'(\tau))^2 + (y'(\tau))^2))^{1/2} d\tau \\ &\leq \sqrt{2}t. \end{aligned}$$

Clearly, $t \leq s(t)$. So

$$t \leq s(t) \leq \sqrt{2}t.$$

Hence (2.28) holds and D with the metric d_D^* is a complete metric space, it is contradiction with the Efimov's Main Lemma. Thus $M_{I\!I\!I}$ is pseudo convex. □

2.4 Proof of Lemma 2.4

Lemma 2.4 *If $i : \Omega \longrightarrow \mathbb{S}^2$ is a C^1 immersion of a surface Ω, inducing a C^0 Riemannian metric on Ω, such that i is an isometric immersion. If Ω is pseudo convex, then*

(A) i is injective on Ω;

(B) $i(\Omega) = \mathbb{S}^2$ or $i(\Omega)$ is convex;

and

(C) Ω has finite area which is equal to 4π if $i(\Omega) = \mathbb{S}^2$, and is no greater than 2π otherwise.

Proof A parametrized arc γ on Ω is called a geodesic if it is locally a shortest arc between any two points on it. i is C^1 and is locally isometric, so if γ is a geodesic arc on Ω, then $i \circ \gamma$ is a great circular arc on $\mathbb{S}^2$, moreover if γ is a geodesic parametrized by its arclength, then also is $i(\gamma)$. Hence, if the length of a geodesic γ on Ω is less than π, then i is injective on γ, and $i \circ \gamma$ is an arc of equal length along a great circle on $\mathbb{S}^2$.

Given any $p \in \Omega$ and $r > 0$, then set $D_r(p) = \{q \in \Omega \mid d(p,q) < r\}$ is called the geodesic disc of radius r about p. Moreover $D_r(p)$ is called a full geodesic disc on Ω if one may leave p in every direction along a (half open) geodesic ray of length r in Ω.

Claim 1 If $D_r(p)$ is a full geodesic disc on Ω with $r > \pi$ then Ω is a sphere, and $i : \Omega \longrightarrow \mathbb{S}^2$ is an isometry. For if $D_r(p)$ is a full geodesic disc on Ω with $r > \pi$ and i is injective on $D_\pi(p)$, so $i(D_\pi(p)) = \mathbb{S}^2 \setminus \{\text{the point } p_0 \text{ antipodal to } i(p)\}$ and $i|_{D_r(p)}$ is an isometry. Let $\partial D_\pi(p) = \{q \in \Omega \mid d(p,q) = \pi\}$, where $d(p,q)$ is the infimum of the lengths with respect to the induced metric by i of the arcs joining p and q in Ω, then $i(\partial D_\pi(p)) = p_0$. Since $\partial D_\pi(p)$ is connected, and i is locally injective, $\partial D_\pi(p)$ must consist of a single point, and Ω must be a sphere, and $i : \Omega \longrightarrow \mathbb{S}^2$ is an isometry.

Claim 2 If $D_r(p)$ is a full geodesic disc on Ω with $r > \pi/2$, then Ω is a sphere and $i : \Omega \longrightarrow \mathbb{S}^2$ is an isometry.

Proof of Claim 2 Given a full geodesic disc $D_r(p)$ on Ω with $r > \pi/2$, if it can be enlarged to be a full geodesic disc of radius larger than π about p, then the conclusion is true. Otherwise, one can assume that $r \leq \pi$ is maximal, then i is injective on $D_r(p)$, so $i(D_r(p))$ is a geodesic disc of radius r on $\mathbb{S}^2$ and there exists $\tilde{q} \in \partial\widetilde{\Omega}$, and $\tilde{q}$ is on the metric closure of $D_r(p)$ in $\widetilde{\Omega}$.

Choose a nongeodesic circle Γ through $\tilde{i}(\tilde{q})$ on $\mathbb{S}^2$, satisfying that the larger of the two regions on $\mathbb{S}^2$ bounded by Γ is contained in $i(D_r(p))$. Take an open arc γ on Γ with $\tilde{i}(\tilde{q}) \in \gamma$ (see Figure 2.21), and let

$$U = i^{-1}(\text{interior}R(\gamma, \sigma))$$

for some small $\sigma > 0$, satisfying that the interior of $R(\gamma, \sigma)$ is in $i(D_r(p))$, then

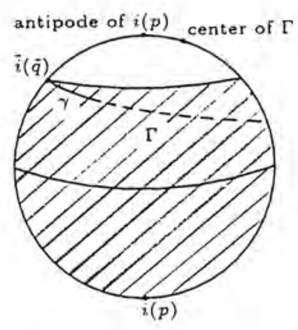

Figure 2.21: $i(D_r(p))$ shaded; $\pi/2 < r < \pi$.

$\tilde{i}$ is injective on $\widetilde{U}$, and $i(U)$ is the interior of $R(\gamma, \sigma)$, so $\widetilde{\Omega}$ is concave at $\tilde{q}$. It is contradiction with the fact that Ω is pseudo convex. Hence, $D_r(p)$ can be enlarged to be a full geodesic disc of radius larger than π, and the conclusion holds. □

Claim 3 If Ω is not a sphere, and γ is a geodesic arc joining two points in Ω, then γ has length $l(\gamma) < \pi$.

Proof of Claim 3 Suppose Ω is not a sphere, and γ is geodesic arc joining two points in Ω with $l(\gamma) \geq \pi$. Replacing γ by a portion of itself if necessay, one can assume $l(\gamma) = \pi$. For simplicity, picture $i(\gamma)$ as parametrzing the portion $y \leq 0$ of the equator on $\mathbb{S}^2$. (By a rotation if necessary)

Let σ-strip around of $i(\gamma)$ be $\cup_{p \in i(\gamma)} D_\sigma(p)$. γ is compact in Ω and i is a C^1 immersion, so there exist $\sigma > 0$ and neighborhood N_0 of γ in Ω, such that $i(N_0)$ is the σ-strip around $i(\gamma)$ and i is injective on N_0. Choose σ small enough, such that $\sigma > \pi/2$, that the closure $\widetilde{N_0}$ of N_0 in $\widetilde{\Omega}$ is the closure $\overline{N_0}$ of N_0 in Ω, that i is injective on $\overline{N_0}$, and that $i(\overline{N_0})$ is the closure of the σ-strip around $i(\gamma)$.

Given some number $\theta \geq 0$, rotate the σ-strip around $i(\gamma)$ on $\mathbb{S}^2$ upward and downward around the x-axis in $\mathbb{R}^3$ through the angle θ. The interior of the set of all points on $\mathbb{S}^2$ reached during the course of these rotations is called the θ-region of $i(\gamma)$ on $\mathbb{S}^2$. Clearly, 0-region of $i(\gamma)$ on $\mathbb{S}^2$ is just the σ-strip of $i(\gamma)$ on $\mathbb{S}^2$, and if θ_1-region$\subset$ θ_2-region, then $0 \leq \theta_1 \leq \theta_2$.

Let Θ be the supremum of all θ values in the interval $[0, \pi/2]$ for which there exists neighborhood N_θ of γ in Ω satisfying that $i(N_\theta) = \theta$-region of $i(\gamma)$ on $\mathbb{S}^2$ and that i is injective on N_θ. If $\Theta > (\pi/2)-\sigma$, then there exists some full geodesic disc of radius larger than $\pi/2$. It is contradiction with Claim 2. If $\Theta \leq (\pi/2)-\sigma$, then N_Θ is in Ω and some point p on the metric closure of N_Θ in $\widetilde{\Omega}$ must be on $\partial\widetilde{\Omega}$. Clearly $p \notin \overline{N_0}$, since $\widetilde{N_0} = \overline{N_0} \subset \Omega$. Hence, $\tilde{i}(p)$ has distance greater than σ from both endpoints of $i(\gamma)$. Thus $\tilde{i}(p)$ is on the portion of the boundary of $i(N_\Theta)$, and $\tilde{i}(p)$ has distance greater than σ from $i(\gamma)$. One can obtain that $\widetilde{\Omega}$ is concave at p, it is contradiction with that Ω is pseudo convex. Hence Claim 3 holds. □

Claim 4 If Ω is not a sphere, and γ is a geodesic arc in Ω from the center of some full geodesic disc $D_r(p)$ on Ω, then

(A) $l(\gamma) + 2r < \pi$;

(B) There is an open convex set H in Ω satisfying $(\gamma \cup D_r(a) \cup D_r(b)) \subset H$;

(C) γ is the unique geodesic in Ω from a to b, and $l(\gamma) = d(a, b)$.

Proof of Claim 4 Clearly, if $a = b$, then the conclusion is true. So one only need to deal with the case $a \neq b$.

If $l(\gamma)+2r \geq \pi$, then γ would be extended to have length π. It is contradiction with Claim 3. So (A) holds.

To check (B). Picture $i(\gamma)$ as parametrizing some portion of the front half $y < 0$ of the equator on $\mathbb{S}^2$, with its midpoint at $(0, -1, 0)$ in $\mathbb{R}^3$, so $i(\gamma \cup D_r(a) \cup D_r(b))$ is in the front hemisphere of $\mathbb{R}^3$(see Figure 2.22). (A) implies that i is injective on $\gamma \cup D_r(a) \cup D_r(b)$.

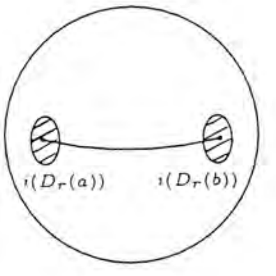

Figure 2.22: $i(D_r(a))$ and $i(D_r(b))$ shaded.

Case 1. $\widetilde{D}_r(a) = \overline{D}_r(a)$, and $\widetilde{D}_r(b) = \overline{D}_r(b)$.

There exists $\sigma > 0$, and a neighborhood $N(\sigma)$ of γ in Ω, such that i maps $N(\sigma)$ one-to-one onto the σ-strip around $i(\gamma)$. Of course, i is still injective on $N(\sigma) \cup \overline{D}_r(a) \cup \overline{D}_r(b)$. Let $\hat{\sigma} > 0$ be the supremum of all σ values in the interval $(0, r)$ for which such a neighborhood $N(\sigma)$ exists, clearly, $N(\hat{\sigma}) \subset \Omega$. To show that $\hat{\sigma} = r$. Suppose $\hat{\sigma} < r$. Then there exists a point $p \in \partial\widetilde{\Omega}$ and $p \in \widetilde{N}(\hat{\sigma})$, moreover $\tilde{i}(p) \in \partial(i(N(\hat{\sigma}))) \setminus \partial(i(D_r(a)) \cup i(D_r(b)))$. It is easily to know that $\widetilde{\Omega}$ would be concave at p (see Figure 2.23). It is contradiction with that Ω is pseudo convex. Hence $\hat{\sigma} = r$.

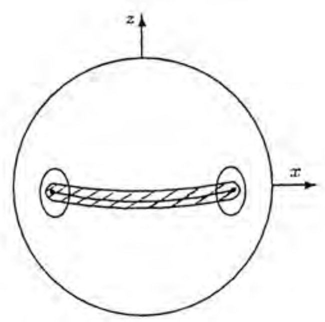

Figure 2.23: $i(N(\hat{\sigma}))$ shaded; $\hat{\sigma} < r$.

To construct such an open convex set H. Let φ be the elliptical cylinder in $\mathbb{R}^3$ formed by all lines parallel to the x-axis through the circular boundaries of both $i(D_r(a))$ and $i(D_r(b))$. Let P_0^+ and P_0^- denote the horizontal planes in $\mathbb{R}^3$ tangent to φ from above and below, respectively. Let $y_0 < 0$ be the y coordinate at the centers of $i(D_r(a))$ and $i(D_r(b))$. $\forall \theta \in (0, \pi/2]$, let P_θ^+ (or P_θ^-) denote the plane which makes an angle θ with the plane $z \equiv 0$ and is tangent to φ along a line on which $y \equiv$ constant $\geq y_0$ and $z =$constant≥ 0 (or $z =$constant≤ 0, respectively). Clearly $P_{\pi/2}^+$ and $P_{\pi/2}^-$ coincide, and are vertical.

There exists a unique $\Theta \in (0, \pi/2]$, such that P_Θ^+ and P_Θ^- hit the origin in $\mathbb{R}^3$, moreover, the planes P_Θ^+ and P_Θ^- cut out great circles on $\mathbb{S}^2$ tangent to the circular boundaries of $i(D_r(a))$ and $i(D_r(b))$. Since (see Figure 2.24), let O, O_1 be the centers in $\mathbb{R}^3$ of $\mathbb{S}^2$ and $\partial(i(D_r(b))) = C_b$, $P_\Theta^+ \cap C_b = A$, and $\overrightarrow{V}$ be

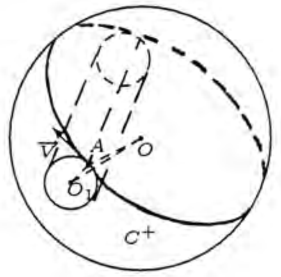

Figure 2.24: $P_\Theta^+ \cap \mathbb{S}^2 = C^+$.

the tangental vector of C_b at A. Then $\vec{V} \in P_\Theta^+$, $\vec{V} \perp \overrightarrow{O_1A}$ and $\vec{V} \perp \overrightarrow{OO_1}$. So $\vec{V} \perp \overrightarrow{OA}$, and $\vec{V}$ is tangent to the great circle $P_\Theta^+ \cap \mathbb{S}^2 = C^+$. Similarly, C^+ is tangent to $\partial(i(D_r(a))) = C_a$, and C_a and C_b are tangent to $P_\Theta^- \cap \mathbb{S}^2$. Let $a_\theta^+ = P_\theta^+ \cap \partial(i(D_r(a)))$, $a_\theta^- = P_\theta^- \cap \partial(i(D_r(a)))$, $b_\theta^+ = P_\theta^+ \cap \partial(i(D_r(b)))$, $b_\theta^- = P_\theta^- \cap \partial(i(D_r(a)))$. Let Γ_θ^a be the arc along the left side of $\partial(i(D_r(a)))$ joining a_θ^+ and a_θ^-, and let Γ_θ^b be the arc along the right side of $\partial(i(D_r(b)))$ joining b_θ^+ and b_θ^-. $\forall\theta \in [0, \Theta]$, let the θ-neighborhood of $i(\gamma)$ on $\mathbb{S}^2$ be the open region on the front face of $\mathbb{S}^2$ bounded by Γ_θ^a, Γ_θ^b, and the front geodesic arcs joining a_θ^+ and b_θ^+, and joining a_θ^- and b_θ^- respectively(see Figure 2.25).

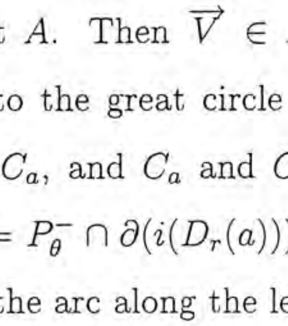

Figure 2.25: θ-neighborhood of $i \circ \gamma$ shaded.

Let θ_1 be the supremum of all $\theta \in [0, \Theta]$ for which there exists a neighborhood H_θ of γ in Ω, such that i is injective on H_θ and $i(H_\theta) = \theta$-neighborhood of $i(\gamma)$ on $\mathbb{S}^2$. Clearly, N_{θ_1} exists. To show that $\theta_1 = \Theta$. Suppose $\theta_1 < \Theta$, then there exists some point $p \in \widetilde{H}_{\theta_1} \cap \partial\widetilde{\Omega}$. $\widetilde{D}_r(a) = \overline{D}_r(a)$ and $\widetilde{D}_r(b) = \overline{D}_r(b)$, so $\tilde{i}(p) \in (P_{\theta_1}^+ \cup P_{\theta_1}^-) \cap \mathbb{S}^2$. Hence $\widetilde{\Omega}$ would be concave at p. It is contradiction with that Ω is pseudo convex. Thus $\theta_1 = \Theta$.

Take $H = H_\Theta$. Then H is convex, since the Θ-neighborhood of $i(\gamma)$ on $\mathbb{S}^2$ is convex, and i is an isometry from H to the Θ-neighborhood of $i(\gamma)$ on $\mathbb{S}^2$.

Case 2. $(\widetilde{D}_r(a) \cup \widetilde{D}_r(b)) \nsubseteq \Omega$.

For any $\rho \in (0, r)$, $D_\rho(a)$ and $D_\rho(b)$ are two full geodesic discs and $\widetilde{D}_\rho(a) = \overline{D}_\rho(a)$, $\widetilde{D}_\rho(b) = \overline{D}_\rho(b)$. From the arguments of Case 1, one obtains an open convex set $H_\rho \supset (\gamma \cup D_r(a) \cup D_r(b))$. Let

$$H = \bigcup_{\rho \in (0,r)} H_\rho,$$

then H is an open convex set, and $H \supset (\gamma \cup D_r(a) \cup D_r(b))$.

Suppose there exist two distinct geodesic arcs γ_1, γ_2 in Ω joining a and b, from Claim 3, if Ω is not a sphere, then $l(\gamma) < \pi$ for any geodesic arc γ in Ω, so $i(\gamma_1) = i(\gamma_2)$, it is contradiction with that i is locally injective on Ω. Of course $l(\gamma) = d(a, b)$.

Hence, the part (C) of Claim 4 is true. $\square$

The End of the Proof of Lemma 2.4 If Ω is a sphere, then i is injective on Ω, $i(\Omega) = \mathbb{S}^2$, and Ω has finite area equal to 4π. So one can assume that Ω is not a sphere.

Suppose p, $q \in \Omega$ with $p \neq q$. Ω is connected, so there exists a parametrized arc Γ in Ω joining p to q. Γ is compact and i is a local isometry, so there exists $\sigma > 0$, such that for any $q_1 \in \Gamma$, $D_\sigma(q_1)$ is a full geodesic disc of radius σ in Ω. Fix a finite set of points $p_0 = p, p_1, p_2, \ldots, p_n = q$ on Γ, indexed in the order indicated by the parametrization of Γ, and satisfying $d(p_{j-1}, p_j) < \sigma$ for any $j = 1, 2, \ldots, n$. Clearly, there exists a unique geodesic γ_1 in Ω joining p_0 to p_1 with $l(\gamma_1) = d(p_0, p_1)$. Suppose that there exists a unique geodesic γ_j in Ω joining p_0 to p_j with $l(\gamma_j) = d(p_0, p_j)$, for any fixed $j = 1, 2, \ldots, n-1$. For $j+1$, since $D_\sigma(p_0)$ and $D_\sigma(p_j)$ are two full geodesic disks with $p_{j+1} \in D_\sigma(p_j)$, and γ_j is a geodesic joining p_0 to p_j, from Claim 4, there exists an open convex set H_j, such that $H_j \supset (\gamma_j \cup D_\sigma(p_0) \cup D_\sigma(p_j))$. So, there exists a unique geodesic γ_{j+1} in

H_j joining p_0 to p_{j+1}, and $l(\gamma_{j+1}) = d(p_0, p_{j+1})$. Note that Claim 3 guarantees $l(\gamma_{j+1}) < \pi$, and that i is a C^1 immersion, then there exists a unique geodesic γ_{j+1} in Ω joining p_0 to p_{j+1}. By induction, there exists a unique geodesic γ_n in Ω joining p to q, with $l(\gamma_n) = d(p, q)$. Thus Ω is convex.

For any $p,\ q \in \Omega$ with $p \neq q$, there exists a unique geodesic γ in Ω joining p to q with $l(\gamma) = d(p, q) < \pi$. Note that i is a C^1 isometric immersion and is injective on γ. So $i(p) \neq i(q)$, i.e., i is injective on Ω.

Clearly, $i(\Omega)$ is convex. Any convex set on $\mathbb{S}^2$ is simply connected and i is an isometry on Ω, so Ω is simply connected.

To prove that if E is a convex subset of $\mathbb{S}^2$, then E is in a hemisphere of $\mathbb{S}^2$. Let q be any point on $\mathbb{S}^2$ whose distance r from $\overline{E}$ is maximal. If $r \geq \pi/2$, then it is true.

If $r = 0$, then E is dense in $\mathbb{S}^2$. $\mathbb{S}^2$ is not convex, so there exists $q \in \mathbb{S}^2 \setminus E$. Let $\mathbb{S}^2_q$ denote the open hemisphere centered at q, and take some geodesic triangle T in $\mathbb{S}^2_q$ whose interior in $\mathbb{S}^2_q$ contains q. E is dense in $\mathbb{S}^2$, so there exist distinct points p_1, p_2 and p_3 in E arbitrarily close to the vertices of T, such that the geodesic triangle determined by the three points has q in its interior in $\mathbb{S}^2_q$ (see Figure 2.26). Clearly, there exists a unique geodesic arc γ in $\mathbb{S}^2_q$ joining p_1 to

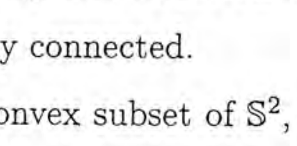

Figure 2.26: Geodesic triangle.

q, extend γ, such that γ hits the side of the geodesic triangle joining p_2 to p_3. E is convex, so the sides of the geodesic triangle must be in E, moreover the extended geodesic arc γ must also be on E. Hence $q \in E$, it is contradiction with $q \in \mathbb{S}^2 \setminus E$.

If $o < r < \pi/2$, then there exists a unique point $p \in \overline{E}$, such that $r = d(q, p)$

$= d(q, \overline{E})$. Suppose there exist p_1, $p_2 \in \overline{E}$ with $p_1 \neq p_2$, such that $d(q, p_1) = d(q, p_2)$, then the shortest geodesic arc on $\mathbb{S}^2$ from p_1 to p_2 is entirely outside of

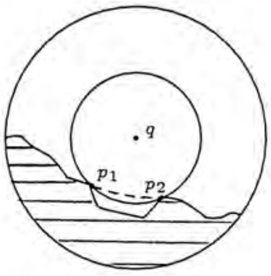

Figure 2.27: $\overline{E}$ shaded.

$\overline{E}$ except at its endpoints. Choose q_1 and q_2 in E arbitarily close to p_1 and p_2, respectively, such that the shortest geodesic arc joining q_1 to q_2 in $\mathbb{S}^2$ has points which are not in $\overline{E}$. (See Figure 2.27.) It is contradiction with that E is convex.

Let U be the geodesic disc of radius $\sigma < r/2$ about p, then the compact set $\overline{E} \setminus U$ has distance $\rho > r$ from q. Move q away from p a distance δ along the great circle through p and q on $\mathbb{S}^2$, where

$$\delta < \min\{\rho - r, \pi/2 - r\},$$

and call the new point reached q_δ. Take midpoint q_m of the shortest geodesic arc γ from q to q_δ, let C_{q_m} be the great circle through q_m making the right angle with γ. (See Figure 2.28.)

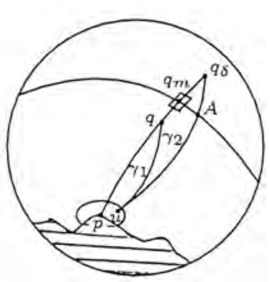

Figure 2.28: $\overline{E}$ shaded.

If $u \in \overline{U}$, then

$$\begin{aligned} d(q_\delta, u) &\leq d(q, q_\delta) + d(q, u) \\ &< \min\{\rho - r, \pi/2 - r\} + d(q, p) + d(p, u) \\ &< \min\{\rho - r, \pi/2 - r\} + r + r/2 \\ &< 3\pi/4 \end{aligned}$$

since $0 < r < \pi/2$. So, there exists an open hemisphere containing $\overline{U}$, q, q_m and q_δ. For any $u \in \overline{U}$, there exists a shortest geodesic arc γ_1 (or γ_2) joining u to q (or q_δ, respectively). Let $A = \gamma_2 \cap C_{q_m}$. Clearly $d(q, A) = d(q_\delta, A)$, so

$$\begin{aligned} d(u, q) &< d(u, A) + d(A, q) \\ &= d(u, A) + d(A, q_\delta) \\ &= d(u, q_\delta). \end{aligned}$$

Note that $\overline{E} \cap \overline{U}$ is compact, and that $d(q, \overline{E} \cap \overline{U}) = r$, so

$$d(q_\delta, \overline{E} \cap \overline{U}) > d(q, \overline{E} \cap \overline{U}) = r.$$

On the other hand, if $u \in \overline{E} \setminus U$, then

$$\begin{aligned} d(q_\delta, u) &\geq d(q, u) - d(q, q_\delta) \\ &> \rho - \delta \\ &> r, \end{aligned}$$

so $d(q_\delta, \overline{E} \setminus U) > r$.

Hence, q_δ is farther from $\overline{E}$ than q is, it is contradiction with the choice of q. Thus $\overline{E}$ must be in a hemisphere on $\mathbb{S}^2$. Clearly Ω has finite area$\leq 2\pi$ if Ω is not a sphere.

Therefore Lemma 2.4 holds. $\square$

Chapter 3

Isometric Immersion into $\mathbb{R}^3$ of Complete Surfaces with Negative Curvature

In this chapter, we always assume that (M, g) is a complete, simply connected, smooth and 2-dimensional surface with negative curvature $-k$, where g is the Riemannian metric on M, and k is some positive smooth function on M. We will discuss the result in Hong [6] first.

Fix a point $p \in M$, and let (e_1, e_2) be the orthonormal basis on T_pM, where T_pM is the tangent space of M at p. By Hardmard Theorem, the exponential map $\exp_p$ is a diffeomorphism from T_pM onto M. Define

$$X = \{q \in M \mid q = \exp_p xe_1, x \in \mathbb{R}\}.$$

Then any point on X is identified by the coordinates $(x, 0)$. One can introduce local geodesic coordinates (x, t) with the base curve X, where t is the oriented distance from the point $(x, 0)$ and $t > 0$ if the point (x, t) is in the image $\{\exp_p(xe_1 + ye_2) \mid y > 0\}$, and $t < 0$ otherwise.

Lemma 3.1 *The geodesic coordinates with the base curve X defined above covers* *the whole manifold M.*

Proof Let $e_2(x)$ be the parallel transport of e_2 along X with $e_2(0) = e_2$. Define $F : \mathbb{R}^2 \longrightarrow M$ by

$$F(x,t) = \exp_{\exp_p xe_1} te_2(x).$$

We want to prove that F is bijective. It is sufficient to prove that for any $p \in M \backslash X$, there exists a unique point $q = X(x_0)$, such that $d(x_0) \leq d(x)$ for all $x \in \mathbb{R}$, where $d(x)$ denotes the distance from $X(x)$ to p, and that the minimizing geodesic $\sigma_0 : [0, d(x_0)] \longrightarrow M$ joining p to q is perpendicular to X. Take a point $q_1 \in X$, then $d(p, q_1) = l < \infty$ for some constant l, and $\overline{B}_l(p)$ is compact. So $\overline{B}_l(p) \cap X$ is compact, hence, there exists a point $q \in X$, such that $d(p,q) = d(p, X)$. Note that M is complete. Hence there exists a minimizing geodesic joining p to q.
To show that $X'(x_0) \perp \sigma_0'(d(x_0))$. Consider the variation $\sigma_s(t), s \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$, $t \in [0, d(x_0)]$, with the variation field $V(t)$ of $\sigma_s(t)$ satisfying $V(0) = 0$ and $V(d(x_0)) = X'(x_0)$, let $E : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$ be the energy of $\sigma_s(t)$. Then

$$\begin{aligned} 0 = \frac{1}{2}E'(0) &= < V(d(x_0)), \frac{\sigma_0}{dt}(d(x_0)) > - < V(0), \frac{\sigma_0}{dt}(0) > \\ &= < X'(x_0), \sigma_0'(d(x_0)) > . \end{aligned}$$

So $X'(x_0) \perp \sigma_0'(d(x_0))$.
To prove the uniqueness. Suppose there exists another point $q' \in X$, such that $d(p,q') = d(p,q)$. From above arguments, the shortest geodesic arc γ from p to q' is perpendicular to X at q'. Let R be the region on M bounded by the portion of X between q' and q, γ and σ_0. Let θ be the external angle made by γ and σ_0(see Figure 3.1). By Gauss-Bonnet Theorem,

$$\int\int_R K d\sigma + \theta + \pi/2 + \pi/2 = 2\pi,$$

where K is the curvature on R, so

$$\int\int_R K d\sigma = \pi - \theta.$$

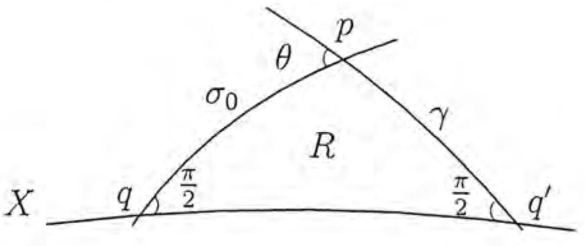

Figure 3.1: Region R.

Note that $\int\int_R K d\sigma < 0$ and $\pi - \theta > 0$, a contradiction is obtained. Hence F is bijective.

To prove that F is locally diffeomorphic. Clearly F is differentiable. Let $F_x = dF(\partial/\partial x)$ and $F_t = dF(\partial/\partial t)$. From the definition of F, one obtains $< F_t, F_t >= 1$. Clearly $< F_t, \nabla_{F_x} F_t >= 0$, so

$$\begin{aligned} \frac{\partial}{\partial t} < F_x, F_t > &= < \nabla_{F_t} F_x, F_t > + < F_x, \nabla_{F_t} F_t > \\ &= < \nabla_{F_t} F_x, F_t > \\ &= 0, \end{aligned}$$

hence $< F_x, F_t >= 0$, thus F_x and F_t form an orthogonal frame on the tangent space. Note that

$$\nabla_{F_t} F_x - \nabla_{F_x} F_t = [F_t, F_x] = dF[\partial/\partial t, \partial/\partial x] = 0, \nabla_{F_t} F_t = 0,$$

and that the curvature of M is negative, then

$$\begin{aligned} \frac{1}{2}\frac{\partial^2}{\partial t^2} < F_x, F_x > &= \frac{\partial}{\partial t} < \nabla_{F_t} F_x, F_x > \\ &= |\nabla_{F_t} F_x|^2 + < \nabla_{F_t} \nabla_{F_t} F_x, F_x > \\ &= |\nabla_{F_t} F_x|^2 + < \nabla_{F_t} \nabla_{F_x} F_t, F_x > \\ &= |\nabla_{F_t} F_x|^2 + < \nabla_{F_t} \nabla_{F_x} F_t - \nabla_{F_x} \nabla_{F_t} F_t + \nabla_{[F_x, F_t]} F_t, F_x > \\ &= |\nabla_{F_t} F_x|^2 + < R(F_x, F_t) F_t, F_x > \\ &= |\nabla_{F_t} F_x|^2 - < R(F_x, F_t) F_t, F_x > \\ &> 0, \end{aligned} \tag{3.1}$$

so $\partial/\partial t < F_x, F_x >$ is strictly increasing. On the other hand,

$$< \nabla_{F_x} F_x, F_t > + < F_x, \nabla_{F_x} F_t >= \frac{\partial}{\partial x} < F_x, F_t >= 0.$$

Hence

$$\begin{aligned} \frac{\partial}{\partial t} < F_x, F_x > &> \frac{\partial}{\partial t} < F_x, F_x > |_{t=0} \\ &= 2 < \nabla_{F_t} F_x, F_x > |_{t=0} \\ &= -2 < \nabla_{F_x} F_x, F_t > |_{t=0} \\ &= 0 \text{ if } t > 0. \end{aligned}$$

Similarly,

$$\frac{\partial}{\partial t} < F_x, F_x >< 0 \text{ if } t < 0.$$

Thus

$$< F_x, F_x > \geq < F_x, F_x > |_{t=0} = 1 \text{ for all } t. \tag{3.2}$$

Therefore the map dF is nonsingular everywhere and F is locally diffeomorphic.

From above arguments, Lemma 3.1 holds. □

Under these geodesic coordinates

$$g = B^2 dx^2 + dt^2, \tag{3.3}$$

where $B = \sqrt{< F_x, F_x >}$.

The following is one of the main results in Hong [6].

Theorem 3.1 (Hong) *(M, g) has a smooth isometric immersion into $\mathbb{R}^3$ if (M, g) with the geodesic coordinates satisfies*

(H_1) $k > 0$ *and* $t\partial_t \ln(k|t|^{2+\delta}) \leq 0$ *as* $|t| \geq T$ *for some positive constants* δ *and* T;

(H_2) $k, \partial_x^i \ln k (i = 1, 2), t\partial_x \partial_t \ln k$ *are bounded, and* $\partial_t^2 \ln k, \partial_t \partial_x \ln k$ *are locally bounded in* t;

(H_3) $\inf\limits_x \int_0^\infty k(x, t) dt$ *and* $\inf\limits_x \int_{-\infty}^0 k(x, t) dt$ *are positive.*

Clearly, one can assume that $0 < \delta < 1$ and $T > 1$.

3.1 The Sketch of the Proof of Theorem 3.1

From Lemma 3.1, it suffices to prove Theorem 3.1 for the surface $(\mathbb{R}^2, g)$ where the curvature $-k$ of g satisfies $(H_1) - (H_3)$. The question of immersion of $(\mathbb{R}^2, g)$ in terms of (x, t) coordinate into $\mathbb{R}^3$ reduces to find the coefficients L, M, and N of the second fundamental form:

$$II = Ldx^2 + 2Mdxdt + Ndt^2,$$

satisfying the Gauss-Codazzi system

$$\begin{cases} L_t - M_x &= L\Gamma_{12}^1 + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N\Gamma_{11}^2, \\ M_t - N_x &= L\Gamma_{22}^1 + M(\Gamma_{22}^2 - \Gamma_{21}^1) - N\Gamma_{21}^2, \\ LN - M^2 &= -kg, \end{cases} \tag{3.4}$$

where g is the determinant of the metric tensor.

Lemma 3.2 *If $-k < 0$ and $L \neq 0$, then (3.4) can be reduced to*

$$\begin{cases} \bar{r}_t + \bar{s}\bar{r}_x &= \frac{1}{2}(\bar{r} - \bar{s})(\widetilde{Q}_t + \bar{r}\widetilde{Q}_x) + A + B\bar{r} + C\bar{s} + D\bar{r}\bar{s} + E\bar{r}^2 + F\bar{r}^2\bar{s}, \\ \bar{s}_t + \bar{r}\bar{s}_x &= \frac{1}{2}(\bar{s} - \bar{r})(\widetilde{Q}_t + \bar{s}\widetilde{Q}_x) + A + B\bar{s} + C\bar{r} + D\bar{s}\bar{r} + E\bar{s}^2 + F\bar{s}^2\bar{r}, \end{cases} \tag{3.5}$$

with $\bar{s} > \bar{r}$, where

$$\widetilde{Q} = \ln(\sqrt{kg}),\ \bar{r} = (-M - \sqrt{kg})/L,\ \bar{s} = (-M + \sqrt{kg})/L,$$
$$A = -\Gamma_{22}^1,\ B = \frac{1}{2}\Gamma_{22}^2 - \frac{3}{2}\Gamma_{12}^1,\ C = \frac{1}{2}(\Gamma_{22}^2 - \Gamma_{21}^1),\ D = \frac{3}{2}\Gamma_{21}^2 - \frac{1}{2}\Gamma_{11}^1,$$
$$E = \frac{1}{2}(\Gamma_{12}^2 - \Gamma_{11}^1),\ F = \Gamma_{11}^2.$$

Proof Clearly,

$$\bar{r}\bar{s} = \frac{M^2 - kg}{L^2},\ \frac{M}{L} = -\frac{\bar{s} + \bar{r}}{2},\ \frac{N}{L} = \bar{r}\bar{s},$$

and

$$\begin{aligned} \frac{1}{2}(\bar{r} - \bar{s})(\widetilde{Q}_t + \bar{r}\widetilde{Q}_x) &= \frac{-\sqrt{kg}}{L}\left\{(\ln\sqrt{kg})_t - \frac{M + \sqrt{kg}}{L}(\ln\sqrt{kg})_x\right\} \\ &= \frac{-\sqrt{kg}}{L}\left\{\frac{(\sqrt{kg})_t}{\sqrt{kg}} - \frac{M + \sqrt{kg}}{L}\frac{(\sqrt{kg})_x}{\sqrt{kg}}\right\} \\ &= -\frac{1}{L^2}\{L(\sqrt{kg})_t - (M + \sqrt{kg})(\sqrt{kg})_x\}. \end{aligned}$$

So

$$
\begin{aligned}
\bar{r}_t + \bar{s}\bar{r}_x &= -\left(\frac{M+\sqrt{kg}}{L}\right)_t - \frac{\sqrt{kg}-M}{L}\left(\frac{M+\sqrt{kg}}{L}\right)_x \\
&= -\frac{LM_t + (\sqrt{kg})_t L - L_t(M+\sqrt{kg})}{L^2} \\
&\quad -\frac{\sqrt{kg}-M}{L}\cdot\frac{LM_x + (\sqrt{kg})_x L - L_x(M+\sqrt{kg})}{L^2} \\
&= -\frac{1}{L^2}\{LM_t + (\sqrt{kg})_t L - L_t(M+\sqrt{kg}) \\
&\quad +\frac{\sqrt{kg}-M}{L}[LM_x + L(\sqrt{kg})_x - L_x(M+\sqrt{kg})]\} \\
&= -\frac{1}{L^2}\{LM_t + (\sqrt{kg})_t L - L_t(M+\sqrt{kg}) + (\sqrt{kg}-M)M_x \\
&\quad +(\sqrt{kg}-M)(\sqrt{kg})_x - \frac{L_x}{L}(kg-M^2)\} \\
&= -\frac{1}{L^2}\{L(\sqrt{kg})_t - (M+\sqrt{kg})(\sqrt{kg})_x + 2\sqrt{kg}(\sqrt{kg})_x \\
&\quad -L_t(M+\sqrt{kg}) + (\sqrt{kg}-M)M_x - \frac{l_x}{L}(kg-M^2) + LM_t\} \\
&= -\frac{1}{L^2}\{L(\sqrt{kg})_t - (M+\sqrt{kg})(\sqrt{kg})_x\} + \frac{L_t}{L}(M+\sqrt{kg}) - \frac{M_t}{L} \\
&\quad -\frac{M_x}{L^2}(\sqrt{kg}-M) + \frac{L_x}{L^3}(kg-M^2) - \frac{2}{L^2}\sqrt{kg}(\sqrt{kg})_x \\
&= \frac{1}{2}(\bar{r}-\bar{s})(\widetilde{Q}_t + \bar{r}\widetilde{Q}_x) - \frac{M_x}{L^2}(\sqrt{kg}-M) + \frac{L_x}{L^3}(kg-M^2) - \frac{M_t}{L} \\
&\quad -\frac{2}{L^2}\sqrt{kg}(\sqrt{kg})_x + \frac{L\Gamma_{12}^1 + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N\Gamma_{11}^2 + M_x}{L^2}(M+\sqrt{kg}) \\
&= \frac{1}{2}(\bar{r}-\bar{s})(\widetilde{Q}_t + \bar{r}\widetilde{Q}_x) - \bar{r}\Gamma_{12}^1 + \frac{\bar{r}(\bar{r}+\bar{s})}{2}(\Gamma_{12}^2 - \Gamma_{11}^1) + \bar{r}^2\bar{s}\Gamma_{11}^2 \\
&\quad +\frac{2MM_x}{L^2} + \frac{L_x(-LN)}{L^3} - \frac{2}{L^2}\sqrt{kg}(\sqrt{kg})_x - \frac{M_t}{L}.
\end{aligned}
$$

$$\begin{aligned}
& \frac{2MM_x}{L^2} + \frac{L_x(-LN)}{L^3} - \frac{2}{L^2}\sqrt{kg}(\sqrt{kg})_x - \frac{M_t}{L} \\
= & \frac{2MM_x - L_xN + (LN - M^2)_x - M^2}{L^2} \\
= & \frac{2MM_x - L_xN + L_xN + LN_x - 2MM_x - LM_t}{L^2} \\
= & \frac{N_x - M_t}{L} \\
= & \frac{-L\Gamma_{22}^1 - M(\Gamma_{22}^2 - \Gamma_{21}^1) + N\Gamma_{21}^2}{L} \\
= & -\Gamma_{22}^1 - \frac{M}{L}(\Gamma_{22}^2 - \Gamma_{21}^1) + \frac{N}{L}\Gamma_{21}^2 \\
= & -\Gamma_{22}^1 + \frac{\bar{r}+\bar{s}}{2}(\Gamma_{22}^2 - \Gamma_{21}^1) + \bar{r}\bar{s}\Gamma_{21}^2,
\end{aligned}$$

so

$$\begin{aligned}
\bar{r}_t + \bar{s}\bar{r}_x = & \frac{1}{2}(\bar{r}-\bar{s})(\widetilde{Q}_t + \bar{r}\widetilde{Q}_x) - \bar{r}\Gamma_{12}^1 + \frac{\bar{r}(\bar{r}+\bar{s})}{2}(\Gamma_{12}^2 - \Gamma_{11}^1) + \bar{r}^2\bar{s}\Gamma_{11}^2 \\
& -\Gamma_{22}^1 + \frac{\bar{r}+\bar{s}}{2}(\Gamma_{22}^2 - \Gamma_{21}^1) + \bar{r}\bar{s}\Gamma_{21}^2 \\
= & \frac{1}{2}(\bar{r}-\bar{s})(\widetilde{Q}_t + \bar{r}\widetilde{Q}_x) - \Gamma_{22}^1 + \bar{r}(\frac{1}{2}\Gamma_{22}^2 - \frac{3}{2}\Gamma_{21}^1) + \frac{\bar{s}}{2}(\Gamma_{22}^2 - \Gamma_{21}^1) \\
& +\bar{r}\bar{s}(\frac{3}{2}\Gamma_{21}^2 - \frac{1}{2}\Gamma_{11}^1) + \frac{\bar{r}^2}{2}(\Gamma_{12}^2 - \Gamma_{11}^1) + \bar{r}^2\bar{s}\Gamma_{11}^2.
\end{aligned}$$

Hence, the first equation holds. Similarly, one can obtain the second equation.

On the other hand. Let

$$L = \frac{2\sqrt{kg}}{\bar{s}-\bar{r}},\ M = \frac{\bar{r}+\bar{s}}{\bar{r}-\bar{s}}\sqrt{kg},\ N = \frac{2\bar{r}\bar{s}}{\bar{s}-\bar{r}}\sqrt{kg},$$

then $LN - M^2 = -kg$ and

$$\begin{aligned}
L_t - M_x &= \left(\frac{2\sqrt{kg}}{\bar{s}-\bar{r}}\right)_t - \left(\frac{\bar{r}+\bar{s}}{\bar{r}-\bar{s}}\sqrt{kg}\right)_x \\
&= -\frac{2\sqrt{kg}(\bar{s}_t - \bar{r}_t)}{(\bar{s}-\bar{r})^2} - \frac{(\bar{r}_x + \bar{s}_x)(\bar{r}-\bar{s}) - (\bar{r}+\bar{s})(\bar{r}_x - \bar{s}_x)}{(\bar{s}-\bar{r})^2}\sqrt{kg} \\
&\quad + \frac{k_t g + k g_t}{\sqrt{kg}(\bar{s}-\bar{r})} - \frac{k_x g + k g_x}{2\sqrt{kg}}\frac{\bar{r}+\bar{s}}{\bar{r}-\bar{s}} \\
&= \frac{\sqrt{kg}(-2\bar{s}_t + 2\bar{r}_t - \bar{r}\bar{r}_x - \bar{r}\bar{s}_x + \bar{s}\bar{r}_x + \bar{s}\bar{s}_x + \bar{r}\bar{r}_x - \bar{r}\bar{s}_x + \bar{s}\bar{r}_x - \bar{s}\bar{s}_x)}{(\bar{s}-\bar{r})^2} \\
&\quad + \frac{k_t g + k g_t}{\sqrt{kg}(\bar{s}-\bar{r})} - \frac{k_x g + k g_x}{2\sqrt{kg}}\frac{\bar{r}+\bar{s}}{\bar{r}-\bar{s}} \\
&= \frac{2\sqrt{kg}(\bar{r}_t + \bar{s}\bar{r}_x - \bar{s}_t - \bar{r}\bar{s}_x)}{(\bar{s}-\bar{r})^2} + \frac{k_t g + k g_t}{\sqrt{kg}(\bar{s}-\bar{r})} - \frac{k_x g + k g_x}{2\sqrt{kg}}\frac{\bar{r}+\bar{s}}{\bar{r}-\bar{s}} \\
&= \frac{2\sqrt{kg}}{(\bar{s}-\bar{r})^2}\big[\frac{1}{2}(\bar{r}-\bar{s})(\widetilde{Q}_t + \bar{r}\widetilde{Q}_x) - \frac{1}{2}(\bar{s}-\bar{r})(\widetilde{Q}_t + \bar{s}\widetilde{Q}_x) \\
&\quad + (B-C)(\bar{r}-\bar{s}) + E(\bar{r}^2 - \bar{s}^2) + F(\bar{r}^2\bar{s} - \bar{s}^2\bar{r})\big] \\
&\quad + \frac{k_t g + k g_t}{\sqrt{kg}(\bar{s}-\bar{r})} - \frac{k_x g + k g_x}{2\sqrt{kg}}\frac{\bar{r}+\bar{s}}{\bar{r}-\bar{s}} \\
&= \frac{2\sqrt{kg}}{(\bar{s}-\bar{r})^2}\big[(\bar{r}-\bar{s})\widetilde{Q}_t + \frac{\bar{r}^2-\bar{s}^2}{2}\widetilde{Q}_x + (B-C)(\bar{r}-\bar{s}) + E(\bar{r}^2-\bar{s}^2) \\
&\quad + F(\bar{r}^2\bar{s} - \bar{s}^2\bar{r})\big] + \frac{k_t g + k g_t}{\sqrt{kg}(\bar{s}-\bar{r})} - \frac{k_x g + k g_x}{2\sqrt{kg}}\frac{\bar{r}+\bar{s}}{\bar{r}-\bar{s}} \\
&= \frac{2\sqrt{kg}}{\bar{s}-\bar{r}}\Gamma^1_{12} - \frac{\bar{s}+\bar{r}}{\bar{s}-\bar{r}}\sqrt{kg}(\Gamma^2_{12} - \Gamma^1_{11}) - \frac{2\bar{r}\bar{s}\sqrt{kg}}{\bar{s}-\bar{r}}\Gamma^2_{11} \\
&= L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11},
\end{aligned}$$

$$\begin{aligned}
M_t - N_x &= \left(\frac{\bar{r}+\bar{s}}{\bar{r}-\bar{s}}\sqrt{kg}\right)_t - \left(\frac{2\bar{r}\bar{s}}{\bar{s}-\bar{r}}\sqrt{kg}\right)_x \\
&= \frac{k_t g + k g_t}{2\sqrt{kg}}\frac{\bar{r}+\bar{s}}{\bar{r}-\bar{s}} - \frac{k_x g + k g_x}{\sqrt{kg}}\frac{\bar{r}\bar{s}}{\bar{s}-\bar{r}} \\
&\quad + \frac{(\bar{r}_t + \bar{s}_t)(\bar{r}-\bar{s}) - (\bar{r}_t - \bar{s}_t)(\bar{r}+\bar{s})}{(\bar{r}-\bar{s})^2}\sqrt{kg} \\
&\quad - \frac{2(\bar{r}_x\bar{s} + \bar{r}\bar{s}_x)(\bar{s}-\bar{r}) - 2\bar{r}\bar{s}(\bar{s}_x - \bar{r}_x)}{(\bar{s}-\bar{r})^2}\sqrt{kg} \\
&= \frac{\bar{r}(\bar{s}_t + \bar{r}\bar{s}_x) - \bar{s}(\bar{r}_t + \bar{s}\bar{r}_x)}{(\bar{r}-\bar{s})^2}2\sqrt{kg} \\
&\quad + \frac{k_t g + k g_t}{2\sqrt{kg}}\frac{\bar{r}+\bar{s}}{\bar{r}-\bar{s}} - \frac{k_x g + k g_x}{\sqrt{kg}}\frac{\bar{r}\bar{s}}{\bar{s}-\bar{r}},
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
& \bar{r}(\bar{s}_t+\bar{r}\bar{s}_x)-\bar{s}(\bar{r}_t+\bar{s}\bar{r}_x) \\
= & \frac{\bar{r}\bar{s}-\bar{r}^2}{2}(\widetilde{Q}_t+\bar{s}\widetilde{Q}_x)-\frac{\bar{s}\bar{r}-\bar{s}^2}{2}(\widetilde{Q}_t+\bar{r}\widetilde{Q}_x) \\
& +A(\bar{r}-\bar{s})+C(\bar{r}^2-\bar{s}^2)+D\bar{r}\bar{s}(\bar{r}-\bar{s})+E\bar{r}\bar{s}(\bar{s}-\bar{r}) \\
= & \frac{\bar{s}^2-\bar{r}^2}{2}\widetilde{Q}_t+(\bar{r}\bar{s}^2-\bar{r}^2\bar{s})\widetilde{Q}_x-\Gamma^1_{22}(\bar{r}-\bar{s}) \\
& +\frac{1}{2}(\Gamma^2_{22}-\Gamma^1_{21})(\bar{r}^2-\bar{s}^2)+\Gamma^2_{21}\bar{r}\bar{s}(\bar{r}-\bar{s}).
\end{aligned} \tag{3.7}$$

Combining (3.6) with (3.7), we can obtain

$$\begin{aligned}
M_t-N_x &= -\frac{2\sqrt{kg}}{\bar{r}-\bar{s}}\Gamma^1_{22}+\frac{\bar{r}+\bar{s}}{\bar{r}-\bar{s}}\sqrt{kg}(\Gamma^2_{22}-\Gamma^1_{21})+\frac{2\bar{r}\bar{s}\sqrt{kg}}{\bar{r}-\bar{s}}\Gamma^2_{21} \\
&= L\Gamma^1_{22}+M(\Gamma^2_{22}-\Gamma^1_{21})-N\Gamma^2_{21}.
\end{aligned}$$

Hence, Lemma 3.2 holds □

Lemma 3.3 *With the geodesic coordinates, (3.4) is equivalent to the following system*

$$L_1(r,s) \overset{def}{=} r_t+\frac{s}{B}r_x-f=0 \tag{3.8}$$

$$L_2(r,s) \overset{def}{=} s_t+\frac{r}{B}s_x-q=0 \tag{3.9}$$

with $s>r$, where $r=B\bar{r}$, $s=B\bar{s}$, $f=-s(1+r^2)\partial_t\ln B+\frac{r-s}{4}(\partial_t+\frac{r}{B}\partial_x)\ln k$ and $q=-r(1+s^2)\partial_t\ln B+\frac{s-r}{4}(\partial_t+\frac{s}{B}\partial_x)\ln k$.

Proof Clearly, with the geodesic coordinates, (3.4) is just the following system

$$\begin{cases}
LN-M^2 &= -kB^2, \\
L_t-M_x &= L\partial_t\ln B+NBB_t-M\partial_x\ln B, \\
M_t-N_x &= -M\partial_t\ln B.
\end{cases}$$

$s>r$, and by Lemma 3.2,

$$\begin{aligned}
\bar{r}_t+\bar{s}\bar{r}_x &= \frac{1}{2}(\bar{r}-\bar{s})(\frac{1}{2}\partial_t\ln k+\partial_t\ln B+\frac{\bar{r}}{2}\partial_x\ln k+\bar{r}\partial_x\ln B)-\frac{3\bar{r}}{2}\partial_t\ln B \\
& -\frac{\bar{s}}{2}\partial_t\ln B-\frac{\bar{r}\bar{s}}{2}\partial_x\ln B-\frac{\bar{r}^2}{2}\partial_x\ln B-\bar{r}^2\bar{s}BB_t \\
&= \frac{\bar{r}-\bar{s}}{4}(\partial_t\ln k+\bar{r}\partial_x\ln k)-(\bar{r}+\bar{s})\partial_t\ln B-\bar{r}^2\bar{s}BB_t-\bar{r}\bar{s}\partial_x\ln B.
\end{aligned}$$

So,

$$\left(\frac{r}{B}\right)_t + \frac{s}{B}\left(\frac{r}{B}\right)_x = \frac{r-s}{4B}(\partial_t \ln k + \frac{r}{B}\partial_x \ln k) - \frac{r+s}{B}\partial_t \ln B - \frac{r^2 s B_t}{B^2} - \frac{rs}{B^2}\partial_x \ln B,$$

i.e.,

$$r_t + \frac{s}{B}r_x - \frac{r-s}{4}(\partial_t \ln k + \frac{r}{B}\partial_x \ln k) + s(1+r^3)\partial_t \ln B = 0.$$

Similarly, (3.9) holds. □

If $(u(x,t), v(x,t)) \in C^1(\mathbb{R}\times[0,T'])$ for some positive constant T' is the solution to the following system.

$$\frac{\partial u}{\partial t} + \lambda_1 \frac{\partial u}{\partial x} = a_{11}u + a_{12}v + R_1 \tag{3.10}$$

$$\frac{\partial v}{\partial t} + \lambda_2 \frac{\partial v}{\partial x} = a_{21}u + a_{22}v + R_2 \tag{3.11}$$

with

$$u(x,0) = \phi_1(x), v(x,0) = \phi_2(x) \text{ and } \lambda_1 \leq \lambda_2, \tag{3.12}$$

where λ_i, a_{ij} and $R_i (i,j = 1,2)$ are C^1 functions of x, t. Then, for any point $(x,t) \in (\mathbb{R}\times[0,T'])$, there exists a unique associated characteristic curve $X = \Gamma_i'(\tau; x, t)$ to λ_i, satisfying

$$\frac{dX}{d\tau} = \lambda_i(X,\tau), \tau < t, \text{ with } X(t) = x.$$

Clearly

$$\Gamma_2'(\tau; x,t) \leq \Gamma_1'(\tau; x,t) \qquad 0 \leq \tau \leq t \leq T'.$$

For any point $A(x^*, t^*) \in \mathbb{R}\times[0,T']$, let

$$\begin{aligned}
\triangle_A' &= \{(x,t) \mid \Gamma_2'(t; x^*, t^*) \leq x \leq \Gamma_1'(t; x^*, t^*), 0 \leq t \leq t^*\}, \\
I(s) &= [\Gamma_2'(s; x^*, t^*), \Gamma_1'(s; x^*, t^*)], \\
H' &= \text{the maximum}\{|a_{ij}|, |\partial_x a_{ij}|, |\partial_x \lambda_i|\} \text{ over } \triangle_A' \text{ and } i,j = 1,2.
\end{aligned}$$

The proof of the Theorem 3.1 follows from the following four lemmas which will be proved later.

Lemma 3.4 *If (u,v) is the C^1 solution in $\triangle'_A(x^*,t^*)$ of (3.10), (3.11) where $R_i = 0$, with (3.12). Then for $(x,t) \in \triangle'_A$*

$$|u|, |v|, |u_x|, |v_x| \le \max_{I(0)} \{ \sum_{i=1}^{i=2} (|\phi_i| + |\partial_x \phi_i|) \} \exp 5H't. \tag{3.13}$$

Lemma 3.5 *Let $(H_1) - (H_3)$ be fulfilled. Then*

$$\frac{B_t}{B} = \frac{1}{t}(1 + O(1/|t|^\delta)) \tag{3.14}$$

uniformly in x for sufficiently large $|t|$ and

$$\partial_t \ln B, \partial_t\partial_t \ln B, B^{1+\delta/2}\partial_t\partial_x \ln B \text{ and } \partial_x \ln B \text{ are bounded .} \tag{3.15}$$

Lemma 3.6 *There exist two positive constants T (big enough) and ψ^* (small enough). Such that, if a C^1 solution (r,s) to the system (3.8), (3.9) in $\triangle_A = \{(x,t) \mid \Gamma_2(t;x^*,t^*) \le x \le \Gamma_1(t;x^*,t^*), T \le t \le t^*\}$ with $s > r$, satisfies*

$$-\psi_0 \le r(x,T) < s(x,T) \le \psi_0 \qquad \forall (x,T) \in \triangle_A \tag{3.16}$$

where $\psi_0 \le \psi^*$, and $\Gamma_i(t) = \Gamma_i(t;x^*,t^*), i = 1,2$, are the characteristic curves corresponding to r/B and s/B respectively, and passing through the point $A(x^*,t^*)$, then

$$-\psi_0 \le r(x,t) < s(x,t) \le \psi_0 \qquad \forall (x,t) \in \triangle_A. \tag{3.17}$$

Lemma 3.7 *There exist smaller ψ^* and bigger T, such that $\mu_* = \inf_{x\in\mathbb{R}} \frac{B_t(x,T)}{2} > 0$, and that, if a C^1 solution (r,s) to the system (3.8), (3.9) in $\triangle_A$ defined in Lemma 3.6 satisfies*

(A) $s > r$ in $\triangle_A$,

(B) for any $x \in [\Gamma_2(T), \Gamma_1(T)]$, $-\mu_* \le r_x(x,T), s_x(x,T)$,
and $-\psi_0 \le r(x,T) < s(x,T) \le \psi_0 \le \psi^*$,

then there exist two positive strictly increasing functions $\theta_i(t)$ defined in $[T,t^*]$ $i = 1,2$, such that, in $\triangle_A$, $|s_x|, |r_x|, |s_t|, |r_t| \le \theta_2(t)$ and

$$\theta_1(t)(s-r) \ge \inf_{x\in[\Gamma_2(T),\Gamma_1(T)]} (s(x,T) - r(x,T)).$$

Here $\theta_i(t)$ depends on the minimum over $[\Gamma_2(T), \Gamma_1(T)]$ of $s - r$ as well as the C^1 bounds of r and s over $[\Gamma_2(T), \Gamma_1(T)]$.

The End of the Proof of Theorem 3.1 Choose T large enough and $\psi*(\leq \frac{1}{16})$ small enough such that Lemma 3.5, Lemma 3.6 and Lemma 3.7 hold. (3.8) and (3.9) comprise a canonical quasi-linear system. It is well known that the local smooth solution to the system exists if the initial data are smooth and bounded. (Cf [[4], p. 121].) Let

$$\begin{aligned} H &= 1 + \sup_{|t| \leq T} \{ |\partial_x \ln B| + 2|\partial_t \ln B| + |\partial_t \ln k| + |\partial_x \ln k| \\ &\quad + 2|\partial_x \partial_t \ln B| + |\partial_x \partial_t \ln k| + |\partial_x^2 \ln k| + |\partial_x \ln B \partial_x \ln k| \}. \end{aligned}$$

Choose ϵ so small that

$$\epsilon \exp 7HT \leq \min\{\psi*, \mu_*\} \leq \frac{1}{16}$$

where $\mu_* = \inf_{x \in \mathbb{R}} \frac{B_t(x,T)}{2}$. Noting that

$$\begin{aligned} &\sup_{|t| \leq T} \Big\{ \Big| \frac{s_x}{B} - \frac{s}{B} \partial_x \ln B \Big|, \ \Big| (1 + r^2) \partial_t \ln B + \frac{1}{4} (\partial_t \ln k + \frac{r}{B} \partial_x \ln k) \Big|, \\ &\quad \Big| (1 + r^2) \partial_x \partial_t \ln B + 2 r r_x \partial_t \ln B + \frac{1}{4} (\partial_x \partial_t \ln k + \frac{r}{B} \partial_x^2 \ln k \\ &\quad + \frac{r_x}{B} \partial_x \ln k - \frac{r}{B} \partial_x \ln B \partial_x \ln k) \Big|, \ \Big| \frac{1}{4} (\partial_t \ln k + \frac{r}{B} \partial_x \ln k) \Big|, \\ &\quad \Big| \frac{1}{4} (\partial_x \partial_t \ln k + \frac{r}{B} \partial_x^2 \ln k + \frac{r_x}{B} \partial_x \ln k - \frac{r}{B} \partial_x \ln B \partial_x \ln k) \Big|, \\ &\quad \Big| \frac{r_x}{B} - \frac{r}{B} \partial_x \ln B \Big|, \Big| (1 + s^2) \partial_t \ln B + \frac{1}{4} (\partial_t \ln k + \frac{s}{B} \partial_x \ln k) \Big| \\ &\quad \Big| (1 + s^2) \partial_x \partial_t \ln B + 2 s s_x \partial_t \ln B + \frac{1}{4} (\partial_x \partial_t \ln k \\ &\quad + \frac{s_x}{B} \partial_x \ln k - \frac{s}{B} \partial_x \ln B \partial_x \ln k + \frac{s}{B} \partial_x^2 \ln k) \Big|, \ \frac{1}{4} \Big| \partial_t \ln k + \frac{s}{B} \partial_x \ln k \Big|, \\ &\quad \frac{1}{4} \Big| \partial_x \partial_t \ln k + \frac{s_x}{B} \partial_x \ln k + \frac{s}{B} \partial_x^2 \ln k - \frac{s}{B} \partial_x \ln B \partial_x \ln k \Big| \Big\} \\ &\leq H \end{aligned}$$

if $|r|, |s|, |r_x|, |s_x| \leq 1$ since the supremum of every term is no more than H, and that $\partial_x \ln k$, $\partial_x^2 \ln k$, $\partial_t \ln B$, $B \partial_t \partial_x \ln B$, $\partial_x \ln B$ are bounded, $\partial_x \partial_t \ln k$, $\partial_t \ln k$

are locally bounded in t. By Lemma 3.4, one obtains that (3.8) and (3.9) can be solved in $\mathbb{R} \times [0, T]$, and that $|\partial_x^i r(x,t)|$, $|\partial_x^i s(x,t)| \le \min\{\psi*, \mu_*\}$ for all $(x,t) \in \mathbb{R} \times [0,T]$, $i = 0$ or 1. Let

$$C(T) = \exp\left[- \left(\sup_{\substack{x\in\mathbb{R}\\ t\le T}} \left|\frac{s_x}{B} + Q\right|\right) T\right]$$

where

$$Q = \frac{q-f}{s-r} = (1-rs)\frac{B_t}{B} + \frac{1}{4}(\partial_\alpha \ln k + \partial_\beta \ln k), \tag{3.18}$$

where $\partial_\alpha = \partial_t + \frac{r}{B}\partial_x$, $\partial_\beta = \partial_t + \frac{s}{B}\partial_x$. To prove that $s - r \ge 2\epsilon C(T)$ for all $(x,t) \in \mathbb{R} \times [0,T]$, it is reduced to prove that for any fixed point (x^*, T), $s - r \ge 2\epsilon C(T)$ for all $(x,t) \in \triangle'_{A(x^*,T)}$. Suppose not, then one can find a point $(x_1, t_1) \in \triangle'_{A(x^*,T)}$ satisfying that $s(x_1,t_1) - r(x_1,t_1) < 2\epsilon C(T)$, and that $s(x,t) - r(x,t) > 0$ for $(x,t) \in \triangle'_{A(x^*,T)} \cap \{(x,t) \mid 0 \le t \le t_1\}$.

$$\begin{aligned} \partial_\beta(s-r) &= s_t + \frac{s}{B}s_x - r_t - \frac{s}{B}r_x \\ &= s_t + \frac{r}{B}s_x - r_t - \frac{s}{B}r_x + \frac{s-r}{B}s_x \\ &= \frac{s-r}{B}s_x + q - f \\ &= \frac{s-r}{B}s_x + Q(s-r), \end{aligned} \tag{3.19}$$

so

$$\partial_\beta \ln(s-r) = \frac{s_x}{B} + Q,$$

so

$$\int_0^{t_1} \partial_\beta \ln(s-r)dt = \int_0^{t_1} (\frac{s_x}{B} + Q)dt,$$

hence

$$\begin{aligned} s(x_1,t_1) - r(x_1,t_1) &= 2\epsilon \exp \int_0^{t_1} (\frac{s_x}{B} + Q)dt \\ &\ge 2\epsilon C(T). \end{aligned}$$

It is contradiction with the choice of the point (x_1, t_1). Thus

$$s - r \geq 2\epsilon C(T) \qquad \forall (x,t) \in \mathbb{R} \times [0,T].$$

Solve again the Cauchy problem for (3.8) (3.9) with the initial data $r(x,T)$ and $s(x,T)$ in the region $t > T$. By Lemma 3.6 and Lemma 3.7, $|r(x,t)|, |s(x,t)| \leq \psi*$, $|r_x(x,t)|$, $|s_x(x,t)|$, $|r_t(x,t)|$ and $|s_t(x,t)| \leq \theta_2(t)$ and $s - r \geq 2\epsilon/\theta_1(t)C(T)$ in the region where r and s are C^1. So one can extend the solution (r, s) to the whole upper plane. Similarly, (3.8) (3.9) can be solved in the other half plane. Hence (3.8) (3.9) with the initial data $r(x,0) = -\epsilon$, $s(x,0) = \epsilon$ can be solved in the whole plane. Moreover, $s - r > 0$ everywhere. By Lemma 3.3, the Gauss-Codazzi system has a global smooth solution. Thus one can find a surface in $\mathbb{R}^3$ with the metric (3.3). Therefore Theorem 3.1 holds. □

3.2 Proof of Lemma 3.4

Suppose that $\triangle'_A, I(s)$ and H' are given in §3.1, then

Lemma 3.4 *If (u,v) is the C^1 solution in $\triangle'_A(x^*,t^*)$ of (3.10), (3.11) where $R_i = 0$, with (3.12). Then for $(x,t) \in \triangle'_A$*

$$|u|, |v|, |u_x|, |v_x| \leq \max_{I(0)} \Big\{ \sum_{i=1}^{i=2} (|\phi_i| + |\partial_x \phi_i|) \Big\} \exp 5H't. \tag{3.20}$$

Proof By (3.10),

$$\begin{aligned} D_t u &= a_{11}u + a_{12}v, \\ D_t u_x &= a_{11x}u + a_{11}u_x + a_{12x}v + a_{12}v_x - \lambda_{1x}u_x. \end{aligned}$$

Integrating them along the characteristic curves, one obtains

$$\begin{aligned} u_x(x,t) - u_x(x,0) &= \int_0^t a_{11x}u + a_{11}u_x + a_{12x}v + a_{12}v_x - \lambda_{1x}u_x, \\ u(x,t) - u(x,0) &= \int_0^t a_{11}u + a_{12}v. \end{aligned}$$

Let $m(s) = \max_{I(s)}\{|u(x,s)|, |v(x,s)|, |\partial_x u(x,s)|, |\partial_x v(x,s)|\}$, then

$$\begin{aligned} |u_x(x,t)| &\leq |u_x(x,0)| + 5H' \int_0^t m(s), \\ |u(x,t)| &\leq |u(x,0)| + 2H' \int_0^t m(s). \end{aligned}$$

Similarly

$$\begin{aligned} |v_x(x,t)| &\leq |v_x(x,0)| + 5H' \int_0^t m(s), \\ |v(x,t)| &\leq |v(x,0)| + 2H' \int_0^t m(s). \end{aligned}$$

So

$$m(t) \leq \max_{I(0)}\{\sum_{i=1}^{i=2}(|\phi_i| + |\partial_x \phi_i|)\} + 5H' \int_0^t m(s).$$

By Gromwall inequality, one obtains (3.20). $\square$

3.3 Proof of Lemma 3.5

Lemma 3.5 *Let $(H_1) - (H_3)$ be fulfilled. Then*

$$\frac{B_t}{B} = \frac{1}{t}(1 + O(1/|t|^\delta)) \tag{3.21}$$

uniformly in x for sufficiently large $|t|$ and

$$\partial_t \ln B, \partial_t\partial_t \ln B, B^{1+\delta/2}\partial_t\partial_x \ln B \text{ and } \partial_x \ln B \text{ are bounded .} \tag{3.22}$$

Proof By (3.2),

$$B^2 =< F_x, F_x > \geq < F_x, F_x > |_{t=0} \text{ for all } (x,t) \in \mathbb{R}^2.$$

By (3.1),

$$2BB_t = \frac{\partial}{\partial t} < F_x, F_x > \geq -2 < \nabla_{F_x} F_x, F_t > |_{t=0} = 0 \text{ for all } (x,t) \in \mathbb{R}^2.$$

So

$$B(x, 0) = 1 \text{ and } B_t(x, 0) = 0 \text{ for all } x \in \mathbb{R}. \tag{3.23}$$

By Gauss equation,

$$B_{tt} = kB. \tag{3.24}$$

(3.24) says that $B_{tt} = kB$, and (3.23) says that $B(x, 0) = 1$ for all $x \in \mathbb{R}$, so

$$\begin{aligned} B_t &= \int_0^t kBds \\ &= -\int_0^t B\frac{d}{ds}\int_s^t kd\xi \\ &= -\big[B\int_s^t kd\xi\big]\big|_0^t + \int_0^t (B_s\int_s^t kd\xi)ds \\ &= \int_0^t kds + \int_0^t (B_s\int_s^t kd\xi)ds \\ &= b_0(t) + \int_0^t B_s N(s, t)ds \end{aligned} \tag{3.25}$$

where

$$b_0(t) = \int_0^t kds \text{ and } N(t, s) = \int_s^t kd\xi.$$

(3.25) is an integral equation of Volterra type (cf [[2], p. 158]), one can find a solution to it in terms of Neumann series

$$B_t = \sum_{n=0}^{\infty} I^n b_0 \tag{3.26}$$

where

$$I^0 b_0 = b_0 \text{ and } I^n b_0 = \int_0^t (I^{n-1}b_0)(s)N(t, s)ds.$$

To prove that

$$I^n b_0 \leq b_0(t)\left[\int_0^t skds\right]^n \frac{1}{n!} \qquad \text{for } n \in \mathbb{N} \cup \{0\} \text{ and } t \geq 0$$

by induction on n. Clearly, it is true if $n = 0$. Suppose that it is true for all $m \le n$. Then, let $f(s) = \int_0^s \xi k d\xi, N = \int_s^t k d\xi$, then

$$N(t, s) \ge 0, N_s(t, s) = -k < 0, f(0) = 0 = N(t, t),$$

and

$$f \ge 0, f'(s) = sk = -sN_s(t, s) \ge 0.$$

So

$$\begin{aligned} \int_0^t f^n N(t, s) ds &= sf^n N|_0^t - \int_0^t s(f^n N)' \\ &= -\int_0^t s(nf^{n-1} f' + f^n N') \\ &\le \int_0^t f^n(-sN') \\ &= \int_0^t f^n f'. \end{aligned}$$

On the other hand, $b_0(s) \ge 0$ is increasing. Hence

$$\begin{aligned} I^{n+1} b_0 &= \int_0^t (I^n b_0)(s) N(t, s) ds \\ &\le \int_0^t b_0(s) \left[\int_0^s \xi k d\xi \right]^n \frac{1}{n!} N(t, s) ds \\ &= \frac{1}{n!} b_0(t) \int_0^t f^n N(t, s) ds \\ &\le \frac{1}{n!} b_0(t) \int_0^t f^n f' ds \\ &= b_0(t) \left[\int_0^t skds \right]^{n+1} \frac{1}{(n+1)!}. \end{aligned}$$

Similarly, one can obtain

$$0 \ge I^n b_0 \ge b_0(t) \left[\int_0^t skds \right]^n \frac{1}{n!} \text{ for } n \in \mathbb{N} \cup \{0\} \text{ and } t < 0.$$

Hence

$$\int_0^t kds \le B_t \le \int_0^t kds \exp\left[\int_0^t skds \right] \text{ for all } t \ge 0, \tag{3.27}$$

and

$$\int_0^t kds \geq B_t \geq \int_0^t kds \exp\left[\int_0^t skds\right] \text{ for all } t < 0. \tag{3.28}$$

One wants to prove that

(H_1') $$0 < k \leq C/(1+|t|)^{2+\delta}$$

for some positive constant C. From (H_1), $t\partial_t \ln(k|t|^{2+\delta}) \leq 0$ as $|t| \geq T$, $T > 1$ and $0 < \delta < 1$.

If $t \leq T$, then $\partial_t \ln(kt^{2+\delta}) \leq 0$. So

$$\int_T^t \partial_s \ln(ks^{2+\delta})ds \leq 0,$$

i.e.,

$$\ln(kt^{2+\delta}) \leq \ln(kT^{2+\delta}).$$

So

$$kt^{2+\delta} \leq e^{\ln(k_1 T^{2+\delta})} = C_0, \text{ for some positive constant } C_0,$$

where $k_1 = \sup\limits_M k$, hence

$$k \leq \frac{C_0}{t^{2+\delta}} = \frac{2C_0}{t^{2+\delta}+t^{2+\delta}} \leq \frac{2C_0}{1+t^{2+\delta}} \leq \frac{C_0}{(\frac{1+t}{2})^{2+\delta}},$$

so there exists $C_1 > 0$, such that $k \leq C_1/(1+t)^{2+\delta}$. Since $k \leq k_1$, let $C_2 = k_1(1+T)^{2+\delta}$, then

$$k \leq C_2/(1+t)^{2+\delta}, 0 \leq t \leq T.$$

Hence, there exists $C_3 > 0$, such that

$$k \leq C_3/(1+t)^{2+\delta} \text{ for all } t \geq 0.$$

Similarly, there exists $C_3' > 0$, such that

$$k \leq C_3'/(1-t)^{2+\delta} \text{ for all } t < 0.$$

Therefore, there exists $C > 0$, such that

$$0 < k \leq C/(1+|t|)^{2+\delta} \text{ for all } t \in \mathbb{R}.$$

By (3.27) and (3.28), one can obtain that

$$1 + \int_0^1 ds \int_0^s k d\xi \leq B.$$

To prove that $B \leq 1 + C_4|t|$ for some positive C_4.

If $t \geq 0$, then $B_t \leq \int_0^t kds \exp[\int_0^t skds]$ and $0 < k \leq C/(1+t)^{2+\delta}$. So

$$\begin{aligned}
& \int_0^t kds \exp\left[\int_0^t skds\right] \\
\leq & \int_0^t \frac{C}{(1+s)^{2+\delta}} ds \exp\left[\int_0^t \frac{Cs}{(1+s)^{2+\delta}} ds\right] \\
= & \left[\frac{-C}{1+\delta}(1+s)^{(-1-\delta)}\Big|_0^t\right] \exp\left[\int_0^t \frac{-C}{(1+s)^{2+\delta}} ds + C\int_0^t \frac{(1+s)}{(1+s)^{2+\delta}} ds\right] \\
= & \left[\frac{C}{1+\delta} - \frac{C}{1+\delta}(1+t)^{-1-\delta}\right] \exp\left[\frac{C}{1+\delta}(1+t)^{-1-\delta} - \frac{C}{1+\delta}\right. \\
& \left. - \frac{C}{\delta}(1+t)^{-\delta} + \frac{C}{\delta}\right] \\
\leq & \frac{C}{1+\delta} \exp\left[\frac{C}{1+\delta} - \frac{C}{1+\delta} + \frac{C}{\delta}\right] = C_4' \text{ for some constant } C_4' > 0.
\end{aligned}$$

Similarly, if $t < 0$, then there exists $C_4'' > 0$, such that $B \leq 1 - C_4''t$. So there exists $C_4 > 0$, such that $B \leq 1 + C_4|t|$.

Hence

$$1 + \int_0^t ds \int_0^s k d\xi \leq B \leq 1 + C_4|t|. \tag{3.29}$$

By $B_{tt} = kB$,

$$\partial_t \partial_t \ln B = \left(\frac{B_t}{B}\right)_t = \frac{B_{tt}B - B_t^2}{B^2} = \frac{kB^2 - B_t^2}{B^2} = k - \left(\frac{B_t}{B}\right)^2.$$

If $t \geq 0$, then $0 \leq B_t/B = \int_0^t (B_s/B)_s ds \leq \int_0^t kds$. Similarly, if $t < 0$, then $-(B_t/B) \leq -\int_0^t kds$. So $|(B_t/B)| \leq |\int_0^t k(x,s)ds| < \infty$, hence $\partial_t \ln B$ is bounded, also is $\partial_t \partial_t \ln B$.

$$B^2 \partial_x \partial_t \ln B = BB_{xt} - B_t B_x$$

and

$$\partial_t(BB_{xt} - B_tB_x) = BB_{xtt} - B_{tt}B_x = B^2\frac{B_{xtt} - kB_x}{B},$$

so

$$\left|\int_0^t B^2\frac{B_{xss} - kB_x}{B}ds\right| \le \left|\int_0^t B^2\left|\frac{B_{xss} - kB_x}{B}\right|ds\right| = \left|\int_0^t |k_x|B^2(x,s)ds\right|,$$

i.e., $|B^2\partial_t\partial_x \ln B - B^2\partial_t\partial_x \ln B|_{t=0}| \le |\int_0^t |k_x|B^2(x,s)ds|$. So

$$|B^{1+\delta/2}\partial_x\partial_t \ln B| \le B^{-1+\delta/2}\left|\int_0^t |k_x|B^2(x,s)ds\right|.$$

From (H_2), $|\partial_x \ln k| \le \alpha$ for some positive constant α, then $|k_x| \le \alpha|k|$. So

$$\begin{aligned} B^{-1+\delta/2}\left|\int_0^t |k_x|B^2(x,s)ds\right| &\le B^{-1+\delta/2}\alpha\left|\int_0^t |k|B^2(x,s)ds\right| \\ &\le B^{-1+\delta/2}\alpha B^{1-\delta/2}\left|\int_0^t |k|B^{1+\delta/2}(x,s)ds\right| \\ &= \alpha\left|\int_0^t kB^{1+\delta/2}(x,s)ds\right|. \end{aligned}$$

Hence

$$|B^{1+\delta/2}\partial_t\partial_x \ln B| \le \alpha\left|\int_0^t kB^{1+\delta/2}(x,s)ds\right| < \infty.$$

And note that $B \ge 1$, and $\alpha_1'|t| \le B \le \alpha_2'|t|$ for $|t|$ large enough, where α_1' and α_2' are two positive constants, then it is easy to obtain that $\partial_x \ln B$ is bounded. From (3.27) and (H_3),

$$\infty > B_t(x,\infty) \ge \inf_x \int_0^t kds$$

and $B \ge \alpha_1 t$ if $t \ge T_1$ for some positive constants α_1 and T_1. Similarly, $B \ge -\alpha_2 t$ if $t \le -T_2$ for some positive constants α_2 and T_2. If $t \ge T_1$. Let $f_1 = B_t/B$, then $\beta_1/t < f_1 < \beta_2/t$ for all $t \ge T_1$, where β_1 and β_2 are some positive constants. Obviously, $f_1' + f_1^2 = k$, so

$$\int_{T_1}^t (\frac{f_1'}{f_1^2} + 1) = \int_{T_1}^t \frac{k}{f_1^2},$$

i.e.,

$$\begin{aligned}\frac{1}{f_1(t)} &= t - T_1 + \frac{1}{f_1(T_1)} - \int_{T_1}^{t} \frac{k}{f_1^2} \\ &= A + t + O(t^{1-\delta}) \text{ for some constant } A.\end{aligned}$$

So,

$$f_1(t) = \frac{1}{t}(1 + O(t^{-\delta})).$$

Similarly,

$$f_1(t) = \frac{1}{t}(1 + O((-t)^{-\delta})) \text{ for all } t \leq -T_2.$$

Hence, $\frac{B_t}{B} = \frac{1}{t}(1 + O(1/|t|^{\delta}))$ uniformly in x for sufficiently large $|t|$. □

The C^0 estimates Define a function h which satisfies

$$h_{tt} = k^* h \text{ with } h(0) = 1 \text{ and } h_t(0) = 0 \tag{3.30}$$

where

$$k^* = C(1 + |t|)^{-2-\delta/2} > k > 0 \tag{3.31}$$

and C is the same as in (H_1'). In a similar argument, since

$$\int_{-\infty}^{\infty} |s| k^* ds < \infty \text{ and } \int_0^{\infty} k^* ds \cdot \int_{-\infty}^{0} k^* ds > 0,$$

one can obtain

$$\begin{aligned}\frac{h_t}{h} &= \frac{1}{t} + O(\frac{1}{|t|^{1+\delta/2}}) \text{ and} \\ a_1|t| \leq \ h &\leq (1 + a_2|t|) \text{ if } |t| \geq T\end{aligned} \tag{3.32}$$

for some constants a_1, a_2 and T.

Lemma 3.8 *If $(H_1) - (H_3)$ are fulfilled and $t > 0$, then*

$$B_t/B < h_t/h \text{ and } B < h. \tag{3.33}$$

Proof By (3.30), one obtains

$$\left(\frac{h_t}{h}\right)_t = \frac{h_{tt}h - h_t^2}{h^2} = k^* - \left(\frac{h_t}{h}\right)^2.$$

On the other hand, $(B_t/B)_t = k - (B_t/B)^2$. So

$$\left(\frac{h_t}{h}\right)_t - \left(\frac{B_t}{B}\right)_t = k^* - k + \left(\frac{B_t}{B}\right)^2 - \left(\frac{h_t}{h}\right)^2.$$

Let $w = h_t/h - B_t/B$, then

$$w_t = k^* - k - w[(B_t/B) + (h_t/h)] \text{ and } w(0) = 0.$$

$w_t(0) > 0$ and $w(0) = 0$, so $w > 0$ for $t > 0$ and small enough. Suppose that $t_0 > 0$ be the first point at which $w = 0$, then $w_t(t_0) \leq 0$. But $w_t(t_0) \geq k^* - k > 0$. Hence $w > 0$ for $t > 0$, i.e., $0 < B_t/B < h_t/h$ for $t > 0$. So $\partial_t \ln B < \partial_t \ln h$ for $t > 0$. Hence $\ln B < \ln h$ for $t > 0$, i.e., $B < h$ for $t > 0$. □

For any constant $\psi_0 > 0$ (or < 0),

$$L_1(\psi_0, \psi_0) = L_2(\psi_0, \psi_0) = \psi_0(1 + \psi_0^2)\frac{B_t}{B} > 0(\text{ or } < 0) \text{ for } t > 0,$$

so (ψ_0, ψ_0) is a supersolution (or subsolution) to (3.8), (3.9). One wants to find another sub-supersolution to (3.8), (3.9). Consider the following equation

$$\phi_t = \phi(1 + \phi^2)\partial_t \ln h + \frac{\phi}{2}\partial_t \ln k^*, t > T \text{ with } \phi(T) = \psi_0 > 0 \tag{3.34}$$

where h and k^* are defined in (3.30) and (3.31), then its solution is

$$\phi = \frac{hb\sqrt{k^*}}{(1 - 2b^2 \int_T^t hh_s k^* ds)^{1/2}} \text{ with } b^2 = \frac{\psi_0^2}{h^2(T)k^*(T)}. \tag{3.35}$$

Indeed, the equation is multiplied by $-2\phi^{-3}$, then

$$-2\phi^{-3}\phi_t = -2\phi^{-2}(1 + \phi^2)\partial_t \ln h - \phi^{-2}\partial_t \ln k^*.$$

Let $z = \phi^{-2}$, then $z(T) = \psi_0^2$ and

$$z' + z(2\partial_t \ln h + \partial_t \ln k^*) = -2\partial_t \ln h.$$

Set $g(t) = 2\partial_t \ln h + \partial_t \ln k^*$ and $f(t) = -2\partial_t \ln h$, then

$$z = \psi_0^{-2} e^{-G(t)} + e^{-G(t)} \int_T^t f(s) e^{G(s)} ds$$

where $G(t) = \int_T^t g(s)ds$, i.e.,

$$z = \psi_0^{-2} h^{-2} k^{*-1} h^2(T) k^*(T) - 2h^{-2} k^{*-1} \int_T^t h h_s k^* ds.$$

So

$$z^{-1} = \frac{\psi_0^2 h^{-2} k^* h^{-2}(T) k^{*-1}(T)}{1 - 2\psi_0^2 h^{-2}(T) k^{*-1}(T) \int_T^t h h_s k^* ds} = \frac{h^2 k^* b^2}{1 - 2b^2 \int_T^t h h_s k^* ds}$$

where $b^2 = \frac{\psi_0^2}{h^2(T)k^*(T)}$. Hence $\phi = \frac{hb\sqrt{k^*}}{(1-2b^2\int_T^t hh_s k^* ds)^{1/2}}$. From (3.31) and (3.32), one can obtain that $\int_T^t h h_s k^* ds < \infty$ for any $t > T$, so ϕ is smooth in $[T, \infty)$ if ψ_0 is small enough.Hence, one can find a small ψ^* and a big T such that $\phi(t) \in C^\infty([T, \infty))$ if $0 < \psi_0 \le \psi^*$. Moreover, $\phi(t)$ is strictly decreasing, since

$$\begin{aligned}
& (1+\phi^2)\partial_t \ln h + 2^{-1}\partial_t \ln k^* \\
= \;& (1+\phi^2)\frac{1}{t} + (1+\phi^2) O\big(\frac{1}{t^{1+\delta/2}}\big) + \frac{1}{2}\partial_t \ln C(1+t)^{-2-\delta/2} \\
= \;& \frac{4+4\phi^2}{4t} - \frac{4+\delta}{4(1+t)} + (1+\phi^2) O\big(\frac{1}{t^{1+\delta/2}}\big) \\
= \;& \frac{-\delta+4\phi^2}{4t} + \big(\frac{\delta}{4t} - \frac{\delta}{4(1+t)}\big) + \big(\frac{-1}{1+t} + \frac{1}{t}\big) + (1+\phi^2) O\big(\frac{1}{t^{1+\delta/2}}\big) \\
= \;& \frac{-\delta+4\phi^2}{4t} + O\big(\frac{1}{t^{1+\delta/2}}\big) \text{ for a big } T \text{ and a small } \psi_0.
\end{aligned}$$

To prove that $L_1(-\phi, \phi) < 0$ and $L_2(-\phi, \phi) > 0$ if $t \ge T$. Using Lemma 3.8, (3.33) and (3.34), one can obtain

$$\begin{aligned}
L_1(-\phi, \phi) &= -\phi_t + \phi(1+\phi^2)\partial_t \ln B + \frac{\phi}{2}\big(\partial_t \ln k - \frac{\phi}{B}\partial_x \ln k\big) \\
&= \phi(1+\phi^2)\partial_t \ln B - \phi(1+\phi^2)\partial_t \ln h - \frac{\phi}{2}\partial_t \ln k^* \\
&\quad + \frac{\phi}{2}\big(\partial_t \ln k - \frac{\phi}{B}\partial_x \ln k\big) \\
&= \phi(1+\phi^2)(\partial_t \ln B - \partial_t \ln h) + \frac{\phi}{2}\big(-\partial_t \ln k^* + \partial_t \ln k - \frac{\phi}{B}\partial_x \ln k\big) \\
&< \frac{\phi}{2}\big(-\partial_t \ln k^* + \partial_t \ln k - \frac{\phi}{B}\partial_x \ln k\big).
\end{aligned}$$

Note that $\partial_x \ln k$ is bounded and that $a_1 t \le B \le a_2 t$ if $t \ge T$ for a big T, then

$$\begin{aligned}
-\partial_t \ln k^* + \partial_t \ln k - \frac{\phi}{B}\partial_x \ln k &= -\partial_t \ln C(1+t)^{-2-\delta/2} + \partial_t \ln k - \frac{\phi}{B}\partial_x \ln k \\
&= \frac{2+\delta/2}{1+t} + \partial_t \ln k - \frac{\phi}{B}\partial_x \ln k \\
&= \frac{2+\delta}{t} - \frac{\delta}{2t} + O(\frac{1}{t^2}) + \partial_t \ln k - \frac{\phi}{B}\partial_x \ln k \\
&= \partial_t \ln(kt^{2+\delta}) - \frac{\delta}{2t} + O(\frac{1}{t^2}) - \frac{\phi}{B}\partial_x \ln k \\
&\le -\frac{\delta}{2t} + O(\frac{1}{t^2}) - \frac{\phi}{B}\partial_x \ln k \\
&\le 0
\end{aligned}$$

if ϕ small enough.

Hence $L_1(-\phi, \phi) < 0$.

$$\begin{aligned}
L_2(-\phi, \phi) &= \phi_t - \phi(1+\phi^2)\partial_t \ln B - \frac{\phi}{2}\partial_t \ln k - \frac{\phi}{2} \cdot \frac{\phi}{B}\partial_x \ln k \\
&> \frac{\delta}{2t} + O(\frac{1}{t^2}) - \frac{\phi}{B}\partial_x \ln k \\
&\ge 0.
\end{aligned}$$

By (3.34),

$$(1+\phi^2)(\partial_t \ln h - \partial_t \ln B) + \frac{1}{2}(\partial_t \ln k^* - \partial_t \ln k - \frac{\phi}{B}\partial_x \ln k) > 0.$$

Let $\xi = \phi$ small enough, then

$$(1+\xi^2)(\partial_t \ln h - \partial_t \ln B) + \frac{1}{2}(\partial_t \ln k^* - \partial_t \ln k - \frac{\xi}{B}\partial_x \ln k) > 0.$$

So

$$\begin{aligned}
&-(1+2\xi^2)\partial_t \ln B - \frac{1}{2}(\partial_t \ln k + \frac{\xi}{B}\partial_x \ln k) \\
&> -\frac{1}{2}\partial_t \ln k^* - (1+\xi^2)\partial_t \ln h - \xi^2 \partial_t \ln B \\
&= \frac{1+\delta/4}{1+t} - \frac{1+2\xi^2}{t} + O(\frac{1}{t^{1+\delta/2}}) \\
&> 0
\end{aligned}$$

if t large enough and ξ small enough.

In the sequel, unless otherwise stated, we always choose T and ψ^* so big and so small respectively that, for $t \geq T$,

$\phi(t)$ is strictly decreasing, $L_1(-\phi, \phi) < 0, L_2(-\phi, \phi) > 0$, for all $0 < \psi_0 \leq \psi^*$,

and

(H_1'') $\qquad -(\frac{1}{2} + \xi^2)\partial_t \ln B - \frac{1}{4}(\partial_t \ln k + \frac{\xi}{B}\partial_x \ln k) > 0$, for all $|\xi| \leq \psi^*$.

3.4 Proof of Lemma 3.6

Lemma 3.6 *There exist two positive constants T (big enough) and ψ^* (small enough). Such that, if (r, s) is a C^1 solution to the system* (3.8), (3.9) *in $\triangle_A = \{(x, t) \mid \Gamma_2(t; x^*, t^*) \leq x \leq \Gamma_1(t; x^*, t^*), T \leq t \leq t^*\}$ with $s > r$, satisfies*

$$-\psi_0 \leq r(x, T) < s(x, T) \leq \psi_0 \qquad \forall (x, T) \in \triangle_A \tag{3.36}$$

where $\psi_0 \leq \psi^*$, and $\Gamma_i(t) = \Gamma_i(t; x^*, t^*), i = 1, 2$, are the characteristic curves corresponding to r/B and s/B respectively, and passing through the point $A(x^*, t^*)$, then

$$-\psi_0 \leq r(x, t) < s(x, t) \leq \psi_0 \qquad \forall (x, t) \in \triangle_A. \tag{3.37}$$

Proof Choose T and ψ^* as above. Let $\bar{r}$ and $\bar{s}$ be two C^1 functions in $\triangle_A$, and let $\partial_\alpha = \partial_t + (r/B)\partial_x$, $\partial_\beta = \partial_t + (s/B)\partial_x$. Then

$$\begin{aligned}\partial_\beta(r - \bar{r}) + \frac{s - \bar{s}}{B}\bar{r}_x &= r_t - \bar{r}_t + \frac{s}{B}r_x - \frac{\bar{s}}{B}\bar{r}_x \\ &= L_1(r, s) - L_1(\bar{r}, \bar{s}) + \underbrace{f(r, s, x, t) - f(\bar{r}, \bar{s}, x, t)}_{I},\end{aligned}$$

let $f_s(u,x,t) = q_r(u,x,t) = -(1+u^2)\frac{B_t}{B} - \frac{1}{4}(\partial_t + \frac{u}{B}\partial_x)\ln k$, then

$$\begin{aligned}
I &= \tilde{s}(1+\tilde{r}^2)\partial_t \ln B - s(1+r^2)\partial_t \ln B \\
&\quad + \frac{r-s}{4}(\partial_t + \frac{r}{B}\partial_x)\ln k + \frac{\tilde{s}-\tilde{r}}{4}(\partial_t + \frac{\tilde{r}}{B}\partial_x)\ln k \\
&= (\tilde{s}-s)(1+\tilde{r}^2)\partial_t \ln B + s(1+\tilde{r}^2)\partial_t \ln B - s(1+r^2)\partial_t \ln B \\
&\quad + \frac{\tilde{s}-s}{4}(\partial_t + \frac{\tilde{r}}{B}\partial_x)\ln k + \frac{s-\tilde{r}}{4}(\partial_t + \frac{\tilde{r}}{B}\partial_x)\ln k + \frac{r-s}{4}(\partial_t + \frac{r}{B}\partial_x)\ln k \\
&= f_s(\tilde{r},x,t)(s-\tilde{s}) + s(\tilde{r}^2 - r^2)\partial_t \ln B + \frac{s}{4B}\partial_x \ln k(\tilde{r}-r) \\
&\quad - \frac{\tilde{r}-r}{4}\partial_t \ln k - \frac{\tilde{r}^2}{4B}\partial_x \ln k + \frac{r^2}{4B}\partial_x \ln k \\
&= f_s(\tilde{r},x,t)(s-\tilde{s}) + b(r-\tilde{r})
\end{aligned}$$

where $b = -[s(\tilde{r}+r)\partial_t \ln B + \frac{s}{4B}\partial_x \ln k - \frac{1}{4}\partial_t \ln k - \frac{\tilde{r}+r}{4B}\partial_x \ln k]$. So

$$\partial_\beta(r-\tilde{r}) + \frac{s-\tilde{s}}{B}\tilde{r}_x = b(r-\tilde{r}) + f_s(\tilde{r},x,t)(s-\tilde{s}) + L_1(r,s) - L_1(\tilde{r},\tilde{s}). \quad (3.38)$$

Similarly,

$$\partial_\alpha(s-\tilde{s}) + \frac{r-\tilde{r}}{B}\tilde{s}_x = b'(s-\tilde{s}) + q_r(\tilde{s},x,t)(r-\tilde{r}) + L_2(r,s) - L_2(\tilde{r},\tilde{s}) \quad (3.39)$$

where b' is some continuous function in $\triangle_A$. To prove that there exists a constant $T_1 > T$, such that

$$-\psi_0 < r(x,t) < s(x,t) < \psi_0 \qquad \forall (x,t) \in \triangle_A \cap \{(x,t) \mid T < t \leq T_1\}.$$

If $s(x,T) = \psi_0, q_r(\psi_0,x,t) \geq 0$. Replace $(\tilde{r},\tilde{s})$ by (ψ_0,ψ_0), then

$$\partial_\alpha(s-\psi_0) = q_r(\psi_0,x,t)(r-\psi_0) - L_2(\psi_0,\psi_0) < 0.$$

If $s(x,T) = \psi_0, q_r(\psi_0,x,t) < 0$. Replace $(\tilde{r},\tilde{s})$ by $(-\psi_0,\psi_0)$, then

$$\partial_\alpha(s-\psi_0) = q_r(\psi_0,x,t)(r+\psi_0) - L_2(-\psi_0,\psi_0) < 0.$$

If $r(x,T) = -\psi_0, f_s(-\psi_0,x,t) \geq 0$. Replace $(\tilde{r},\tilde{s})$ by $(-\psi_0,-\psi_0)$, then

$$\partial_\beta(r+\psi_0) = f_s(-\psi_0,x,t)(s+\psi_0) - L_1(-\psi_0,-\psi_0) > 0.$$

If $r(x,T) = -\psi_0, f_s(-\psi_0, x, t) < 0$. Replace $(\tilde{r}, \tilde{s})$ by $(-\psi_0, \psi_0)$, then

$$\partial_\beta(r + \psi_0) = f_s(-\psi_0, x, t)(s - \psi_0) - L_1(-\psi_0, \psi_0) > 0.$$

So such T_1 exists.

Suppose that there exists a point (x_1, t_1), such that $r(x_1, t_1) = -\psi_0$ or $s(x_1, t_1) = \psi_0$, and (3.37) holds on $\triangle_A \cap \{(x,t) \mid T < t \le t_1\}$. Then, replace the same argument at $t = t_1$ as done at $t = T$, one obtains that $-\psi_0 < r(x,t) < s(x,t) < \psi_0$ near $t = t_1$ and $t > t_1$.

Therefore Lemma 3.6 holds. □

Suppose that (r, s) is a solution to (3.8), (3.9) and that $\partial_\alpha = \partial_t + \frac{r}{B}\partial_x$, $\partial_\beta = \partial_t + \frac{s}{B}\partial_x$. Then, from (3.19),

$$\partial_\beta(s - r) = \frac{s-r}{B}s_x + Q(s - r), \tag{3.40}$$

where Q is described as (3.18). Similarly,

$$\partial_\alpha(s - r) = \frac{s-r}{B}r_x + Q(s - r). \tag{3.41}$$

Let $\tilde{s} = (s-r)s_x/B, \tilde{r} = (s-r)r_x/B$, where (r, s) is a solution to (3.8), (3.9), then

$$\widetilde{L}_1(\tilde{r}, \tilde{s}) \stackrel{\text{def}}{=} \partial_\beta \tilde{r} - (Q + f_r - \frac{B_t}{B})\tilde{r} - f_s\tilde{s} - \frac{s-r}{B}\delta_x f = 0 \tag{3.42}$$

and

$$\widetilde{L}_2(\tilde{r}, \tilde{s}) \stackrel{\text{def}}{=} \partial_\alpha \tilde{s} - (Q + q_s - \frac{B_t}{B})\tilde{s} - q_r\tilde{r} - \frac{s-r}{B}\delta_x q = 0 \tag{3.43}$$

where δ_x denotes the differentiation only with respect to x. Indeed

$$\begin{aligned}
\widetilde{L}_1(\tilde{r},\tilde{s}) &= \partial_\beta(\frac{s-r}{B}r_x) - (Q+f_r-\frac{B_t}{B})(\frac{s-r}{B}r_x) - f_s(\frac{s-r}{B}s_x) - \frac{s-r}{B}\delta_x f \\
&= \frac{r_x}{B}\partial_\beta(s-r) + (s-r)\partial_\beta(\frac{r_x}{B}) - (Q+f_r-\frac{B_t}{B})(\frac{s-r}{B}r_x) \\
&\quad -f_s(\frac{s-r}{B}s_x) - \frac{s-r}{B}\delta_x f \\
&= \frac{r_x}{B}\partial_\beta(s-r) - \frac{r_x}{B}Q(s-r) + (s-r)\partial_\beta(\frac{r_x}{B}) \\
&\quad -(f_r-\frac{B_t}{B})(\frac{s-r}{B}r_x) - f_s(\frac{s-r}{B}s_x) - \frac{s-r}{B}\delta_x f \\
&= \frac{s-r}{B}[\frac{r_x}{B}s_x + s(\frac{r_x}{B})_x + B(\frac{r_x}{B})_t - f_r r_x + \frac{B_t}{B}r_x - f_s s_x - \delta_x f] \\
&= \frac{s-r}{B}[(\frac{s}{B}r_x)_x + B(\frac{r_x}{B})_t - f_r r_x - f_s s_x - \delta_x f + \frac{B_t}{B}r_x] \\
&= \frac{s-r}{B}[(\frac{s}{B}r_x)_x + r_{xt} - (f)_x] \\
&= 0,
\end{aligned}$$

similarly

$$\widetilde{L}_2(\tilde{r},\tilde{s}) = 0.$$

3.5 Proof of Lemma 3.7

Lemma 3.7 *There exist smaller ψ^* and bigger T, such that $\mu_* = \inf\limits_{x\in\mathbb{R}} \frac{B_t(x,T)}{2} > 0$, and that, if a C^1 solution (r,s) to the system (3.8), (3.9) in $\triangle_A$ defined in Lemma 3.6 satisfies*

(A) $s > r$ in $\triangle_A$,

(B) for any $x \in [\Gamma_2(T), \Gamma_1(T)]$, $-\mu_* \le r_x(x,T), s_x(x,T)$,

and $-\psi_0 \le r(x,T) < s(x,T) \le \psi_0 \le \psi^*$,

then there exist two positive strictly increasing functions $\theta_i(t)$ defined in $[T,t^*]$ $i = 1,2$, such that, in $\triangle_A$, $|s_x|, |r_x|, |s_t|, |r_t| \le \theta_2(t)$ and

$$\theta_1(t)(s-r) \ge \inf_{x\in[\Gamma_2(T),\Gamma_1(T)]}(s(x,T) - r(x,T)).$$

Here $\theta_i(t)$ depends on the minimum over $[\Gamma_2(T), \Gamma_1(T)]$ of $s-r$ as well as the C^1 bounds of r and s over $[\Gamma_2(T), \Gamma_1(T)]$.

Proof We will discuss the following lemma first.

Lemma 3.9 *Let (u, v) be a C^1 solution to the system (3.21)(3.22). Then for any fixed $A(x^*, t^*) \in \mathbb{R} \times [0, T]$, $u(x,t), v(x,t) > 0 (< 0)$ for all $(x,t) \in \triangle_A \cap \{t > 0\}$ if*

(A) $a_{12}(x,t)$ *and* $a_{21}(x,t) \geq 0$ *on* $\triangle_A$;

(B) $\phi_i(\xi) \geq 0 (\leq 0)$ *for all* $\xi \in [\Gamma_2(0; x^*, t^*), \Gamma_1(0; x^*, t^*)], i = 1, 2$;

(C) $R_i(x,t) > 0 (< 0)$ *on* $\triangle_A, i = 1, 2$.

Proof of Lemma 3.9 One can assume that $R_i > 0, \phi_i \geq 0, i = 1, 2$, since a transformation $u \mapsto -u, v \mapsto -v$, reduces the negative case to the positive case. If $u(x_0, 0) = 0$ for $(x_0, 0) \in \triangle_A$, then $\frac{\partial u}{\partial t} + \lambda_1 \frac{\partial u}{\partial x} = a_{12} v + R_1 > 0$ at $(x_0, 0)$. u is C^1, so there exists $t_1 > 0$, such that $u(x,t) > 0$ for $(x,t) \in \triangle_A \cap \{0 < t \leq t_1\}$. Similarly, there exists $t_2 > 0$, such that $v(x,t) > 0$ for $(x,t) \in \triangle_A \cap \{0 < t \leq t_2\}$. Hence, there exists $T_1 > 0$, such that $u(x,t), v(x,t) > 0$ for $(x,t) \in \triangle_A \cap \{0 < t \leq T_1\}$.

Suppose there exists $t_0 \in (0, t^*]$, such that $u(x_0, t_0) = 0$ or $v(x_0, t_0) = 0$ for some $x_0 \in [\Gamma_2(t_0; x^*, t^*), \Gamma_1(t_0; x^*, t^*)]$, and $u(x,t), v(x,t) > 0$ for $(x,t) \in \triangle_{A(x^*,t^*)} \cap \{t < t_0\}$. By a similar argument, one can know that $\frac{\partial u}{\partial t} + \lambda_1 \frac{\partial u}{\partial x} > 0$ or $\frac{\partial v}{\partial t} + \lambda_2 \frac{\partial v}{\partial x} > 0$ at (x_0, t_0), respectively. It is impossible since $u(x,t), v(x,t) > 0$ for $(x,t) \in \triangle_{A(x^*,t^*)} \cap \{t < t_0\}$ and $u(x_0, t_0) = 0$ or $v(x_0, t_0) = 0$. □

Let $w_1 = (s-r)(r_x + \frac{1}{2}B_t)/B$ and $w_2 = (s-r)(s_x + \frac{1}{2}B_t)/B$, then

$$\partial_\beta w_1 = (Q + f_r - \frac{B_t}{B})w_1 + (f_s + \frac{B_t}{2B})w_2 + R_1 \tag{3.44}$$

$$\partial_\alpha w_2 = (q_r + \frac{B_t}{2B})w_1 + (Q + q_s - \frac{B_t}{B})w_2 + R_2 \tag{3.45}$$

where

$$R_1 = \frac{s-r}{B}\{\frac{1}{2}[\frac{B_t^2}{B}(\frac{1}{2} + 2rs + r^2) + Bk + s\partial_t\partial_x \ln B + \frac{s-r}{4}\frac{B_t}{B}\partial_x \ln k] + \delta_x f\}$$

and

$$R_2 = \frac{s-r}{B}\{\frac{1}{2}[\frac{B_t^2}{B}(\frac{1}{2}+2rs+s^2)+Bk+r\partial_t\partial_x \ln B - \frac{s-r}{4}\frac{B_t}{B}\partial_x \ln k] + \delta_x q\}.$$

Indeed

$$\begin{aligned}
\partial_\beta w_1 &= \partial_\beta(\frac{s-r}{B}r_x) + \partial_\beta(\frac{s-r}{2B}B_t) \\
&= (Q+f_r-\frac{B_t}{B})(\frac{s-r}{B}r_x) + f_s(\frac{s-r}{B}s_x) + \frac{s-r}{B}\delta_x f \\
&\quad + \frac{B_t}{2B}\partial_\beta(s-r) + \frac{s-r}{2}\partial_\beta(\frac{B_t}{B}) \text{ by (3.34)} \\
&= (Q+f_r-\frac{B_t}{B})(\frac{s-r}{B}r_x) + f_s(\frac{s-r}{B}s_x) + \frac{s-r}{B}\delta_x f + \frac{B_t}{2B}\cdot\frac{s-r}{B}s_x \\
&\quad + \frac{1}{2}\frac{B_t}{B}Q(s-r) + \frac{s-r}{2}(k - \frac{B_t^2}{B} + \frac{s}{B}\partial_t\partial_x \ln B) \text{ by (3.31)} \\
&= (Q+f_r-\frac{B_t}{B})w_1 + (f_s+\frac{B_t}{2B})w_2 - \frac{B_t}{2B}(s-r)(f_s+f_r) \\
&\quad - \frac{B_t^2}{4B^2}(s-r) + \frac{s-r}{2}k + \frac{s-r}{B}\delta_x f + \frac{s(s-r)}{2B}\partial_t\partial_x \ln B \\
&= (Q+f_r-\frac{B_t}{B})w_1 + (f_s+\frac{B_t}{2B})w_2 + \frac{s-r}{B}\{\frac{1}{2}[\frac{B_t^2}{B}(\frac{1}{2}+2rs+r^2) \\
&\quad + Bk + s\partial_t\partial_x \ln B + \frac{s-r}{4}\frac{B_t}{B}\partial_x \ln k] + \delta_x f\},
\end{aligned}$$

similarly, (3.45) is valid.

To prove $R_1 > 0$ if ψ^* is small enough and T is big enough.

$$\begin{aligned}
&\frac{1}{2}[\frac{B_t^2}{B}(\frac{1}{2}+2rs+r^2) + Bk + s\partial_t\partial_x \ln B + \frac{s-r}{4}\frac{B_t}{B}\partial_x \ln k] + \delta_x f \\
= &\frac{1}{2}\frac{B_t^2}{B}(\frac{1}{2}+2rs+r^2) + \frac{Bk}{2} + \frac{s}{2B}B\partial_t\partial_x \ln B + \frac{s-r}{8}\frac{B_t}{B}\partial_x \ln k \\
&- \frac{s(1+r^2)}{B}B\partial_t\partial_x \ln B + \frac{s-r}{4t}t\partial_t\partial_x \ln k + \frac{r(r-s)}{4B}\partial_x\partial_x \ln k \\
&- \frac{r(r-s)}{4B}\partial_x \ln B \partial_x \ln k.
\end{aligned}$$

Using Lemma 3.5, Lemma 3.6 and (H_2), one can see that the right side of the above equation is positive if ψ^* is small enough and T is big enough, so $R_1 > 0$. Similarly, $R_2 > 0$ if ψ^* is small enough and T is big enough.

To prove that $w_i \geq 0,\ i = 1, 2$. From (H_1''),

$$-(\frac{1}{2}+\xi^2)\partial_t \ln B - \frac{1}{4}(\partial_t \ln k + \frac{\xi}{B}\partial_x \ln k) > 0 \text{ if } t > T \text{ for all } |\xi| \leq \psi^*,$$

so

$$\begin{aligned} f_s + \frac{B_t}{2B} &= -(1+r^2)\partial_t \ln B - \frac{1}{4}(\partial_t + \frac{r}{B}\partial_x)\ln k + \frac{B_t}{2B} \\ &= -(\frac{1}{2}+r^2)\partial_t \ln B - \frac{1}{4}(\partial_t + \frac{r}{B}\partial_x)\ln k \\ &> 0 \text{ if } r \text{ is small enough.} \end{aligned}$$

Similarly $q_r + \frac{B_t}{2B} > 0$ if s is small enough. In addition, by assumption, $s_x(x,T)$ and $r_x(x,T) \geq -\mu_*$, so $w_1(x,T), w_2(x,T) \geq 0$. Using Lemma 3.9, one obtains that $w_i \geq 0$ $i = 1, 2$. $s > r$, so

$$r_x, s_x \geq -\frac{B_t}{2} \text{ on } \triangle_A.$$

To construct $\theta_1(t)$. Note that $s > r$, and that $s_x \geq -\frac{B_t}{2}$, then

$$\begin{aligned} \partial_\beta(s-r) &= \frac{s-r}{B}s_x + Q(s-r) \\ &\geq Q(s-r) - \frac{s-r}{2}\partial_t \ln B. \end{aligned}$$

So

$$\partial_\beta \ln(s-r) \geq Q - \frac{1}{2}\partial_t \ln B,$$

so,

$$\ln(s-r) - \ln(s(T)-r(T)) \geq \int_T^t (Q - \frac{1}{2}\partial_\tau \ln B(x,\tau))d\tau.$$

Using Lemma 3.5, Lemma 3.6 and (H_2), one obtains

$$\begin{aligned} Q - \frac{1}{2}\partial_\tau \ln B(x,\tau) &= (1-rs)\partial_\tau \ln B + \frac{1}{4}(\partial_\alpha \ln k + \partial_\beta \ln k) - \frac{1}{2}\partial_\tau \ln B \\ &= \frac{1}{2}\partial_\tau \ln B - rs\partial_\tau \ln B + \frac{1}{4}(2\partial_\tau \ln k + \frac{r+s}{B}\partial_x \ln k) \\ &= \frac{1}{2}\partial_\tau \ln B - rs\partial_\tau \ln B + \frac{1}{2}\partial_\tau \ln k + \frac{r+s}{4B}\partial_x \ln k \\ &\geq -(A + A/\tau), \end{aligned}$$

where A is a positive constant dependent on T and t^*. So

$$\begin{aligned} \ln(s-r) &\geq \ln(s(T)-r(T)) + \int_T^t (-A - A/\tau)d\tau \\ &= \ln(s(T)-r(T)) - A(t-T) - A\ln(\frac{t}{T}). \end{aligned}$$

Hence

$$\begin{aligned} s-r &\geq \exp[\ln(s(T)-r(T)) - A(t-T) - A\ln(\frac{t}{T})] \\ &\geq \inf_{x\in[\Gamma_2(T),\Gamma_1(T)]}(s(T)-r(T))\exp[-A(t-T) - A\ln(\frac{t}{T})]. \end{aligned}$$

Let $\theta_1(t) = \exp[A(t-T) + A\ln(\frac{t}{T})]$, then

$$\theta_1(t)(s-r) \geq \inf_{x\in[\Gamma_2(T),\Gamma_1(T)]}(s(T)-r(T)). \tag{3.46}$$

To construct $\theta_2(t)$. Note that (3.8) and (3.9) guarantee

$$\begin{aligned} \partial_\beta\partial_\alpha r &= \partial_\alpha\partial_\beta r + [\partial_\beta, \partial_\alpha]r \\ &= \partial_\alpha f + (\partial_\beta(\frac{r}{B})\partial_x - \partial_\alpha(\frac{s}{B})\partial_x)r \\ &= \delta_\alpha f + f_r\partial_\alpha r + f_s\partial_\alpha s + \frac{B\partial_\beta r - r\partial_\beta B - B\partial_\alpha s + s\partial_\alpha B}{B^2}r_x \\ &= \delta_\alpha f + f_r\partial_\alpha r + qf_s + \frac{B(f-q) - rB_t + sB_t}{B^2}r_x \\ &= \delta_\alpha f + f_r\partial_\alpha r + qf_s - (Q - \partial_t\ln B)(s-r)r_x/B \\ &= [f_r + (Q - \partial_t\ln B)]\partial_\alpha r + [\delta_\alpha f + qf_s - (Q - \partial_t\ln B)f] \end{aligned}$$

where $\delta_\alpha f = \delta_x f\partial_\alpha x + \delta_t f\partial_\alpha t = \frac{r}{B}\delta_x f + \delta_t f$,

and that

$$\begin{aligned} q &= -r(1+s^2)\partial_t\ln B + \frac{s-r}{4}(\partial_t + \frac{s}{B}\partial_x)\ln k \\ f &= -s(1+r^2)\partial_t\ln B + \frac{r-s}{4}(\partial_t + \frac{r}{B}\partial_x)\ln k \\ f_r &= -2rs\partial_t\ln B + \frac{1}{4}\partial_t\ln k + \frac{2r-s}{4B}\partial_x\ln k \\ f_s &= -(1+r^2)\partial_t\ln B - \frac{1}{4}(\partial_t + \frac{r}{B}\partial_x)\ln k \\ Q &= (1-rs)\partial_t\ln B + \frac{1}{4}(\partial_\alpha\ln k + \partial_\beta\ln k) \\ \delta_x f &= -s(1+r^2)\partial_t\partial_x\ln B + \frac{r-s}{4}\partial_t\partial_x\ln k + \frac{r(r-s)}{4B}\partial_x^2\ln k \\ &\quad -\frac{r(r-s)}{4B}\partial_x\ln B\partial_x\ln k \\ \delta_t f &= -s(1+r^2)\partial_t^2\ln B + \frac{r-s}{4}\partial_t^2\ln k + \frac{r(r-s)}{4B}\partial_t\partial_x\ln k \\ &\quad -\frac{r(r-s)}{4B}\partial_t\ln B\partial_x\ln k. \end{aligned}$$

so, from Lemma 3.5, Lemma 3.6 and (H_2), one obtains that $f_r + Q - \partial_t \ln B$ and $\delta_\alpha f + qf_s - (Q - \partial_t \ln B)f$ are bounded as $t \in [T, t^*]$. Let $f_1(x,t) = f_r + Q - \partial_t \ln B$ and $f_2(x,t) = \delta_\alpha f + qf_s - (Q - \partial_t \ln B)f$, then $|f_1| \le C_1$ and $|f_2| \le C_2$ for $t \in [T, t^*]$, where C_1, C_2 are two positive constants dependent on T and t^*.

$$\int_T^t \partial_\beta \partial_\alpha r = \int_T^t f_1 \partial_\alpha r + \int_T^t f_2,$$

so

$$\partial_\alpha r(t) - \partial_\alpha r(T) = \int_T^t f_1 \partial_\alpha r + \int_T^t f_2,$$

so

$$\begin{aligned} |\partial_\alpha r| &\le |\partial_\alpha r(T)| + |\int_T^t f_1 \partial_\alpha r| + |\int_T^t f_2| \\ &\le |\partial_\alpha r(T)| + \int_T^t |f_2| + \int_T^t |f_1||\partial_\alpha r| \\ &\le C(T, t^*) + C'(T, t^*) \int_T^t |\partial_\alpha r| \end{aligned}$$

where $C(T,t^*)$ and $C'(T,t^*)$ are two positive constants dependent on T and t^*, by Gronwall inequality,

$$|\partial_\alpha r| \le C(T, t^*) e^{C'(T,t^*)t}.$$

$$r_x = \frac{(\partial_\beta r - \partial_\alpha r)B}{s - r} = \frac{(f - \partial_\alpha r)B}{s - r},$$

so, from (3.46)

$$|r_x| \le \frac{(|f| + |\partial_\alpha r|)B}{s - r} \le \frac{(C_0(T,t^*) + C(T,t^*)e^{C'(T,t^*)t})B}{\inf\limits_{x \in [\Gamma_2(T), \Gamma_1(T)]} (s(x,T) - r(x,T))} \theta_1(t),$$

where $C_0(T,t^*)$ is some positive constant dependent on T and t^*. Let

$$g_1(t) = \frac{(C_0(T,t^*) + C(T,t^*)e^{C'(T,t^*)t})\theta_1(t)}{\inf\limits_{x \in [\Gamma_2(T), \Gamma_1(T)]} (s(x,T) - r(x,T))},$$

then $|r_x| \leq g_1(t)B$. On the other hand, $r_t = f - (sr_x/B)$, so

$$|r_t| \leq |f| + |\frac{r_x}{B}s| \leq C_0(T, t^*) + \psi_0 g_1(t).$$

Let $\theta_2'(t) = C_0(T, t^*) + (\psi_0 + B)g_1(t)$, then $\theta_2'(t)$ is strictly increasing and

$$|r_x|, |r_t| \leq \theta_2'(t) \text{ in } \triangle_A.$$

Similarly, there exists a strictly increasing function $\theta_2''(t)$ defined in $[T, t^*]$, satisfying that

$$|s_x|, |s_t| \leq \theta_2''(t) \text{ in } \triangle_A.$$

Let $\theta(t) = \max\{\theta_2'(t), \theta_2''(t)\}$, then

$$|r_x|, |r_t|, |s_x|, |s_t| \leq \theta(t) \text{ in } \triangle_A. \ \square$$

3.6 The Geometric Properties of the Immersion

In this section, we will discuss some results in Chan [[1], pp. 49-57].

Theorem 3.2 *For the isometric immersion of the surface M with the metric, $ds^2 = B^2(x,t)dx^2 + dt^2$ constructed in Theorem 3.1, the base curve X defined in the beginning of Chapter 3 is a locally convex plane curve.*

Proof By Hong's Theorem, there are solutions r, s to the system (3.8), (3.9), then

$$L = \frac{2B^2\sqrt{k}}{s-r}, \ M = -\frac{(s+r)B\sqrt{k}}{s-r}, \text{ and } N = \frac{2rs\sqrt{k}}{s-r}$$

are the coefficients of the second fundamental form. By Bonnet's fundamental theorem of surface, there is an immersion $V : U \subset \mathbb{R}^2 \longrightarrow V(U) \subset \mathbb{R}^3$. Let

$\{V_x, V_t, \mathcal{N}\}$ be the orthonormal basis, where $\mathcal{N}$ is the unit normal vector on the surface. $E = B^2(x,t)$, $G = 1, F = 0$, so

$$\begin{cases} V_{xx} &= \frac{B_x}{B}V_x - BB_tV_t + L\mathcal{N} \\ V_{xt} &= \frac{B_t}{B}V_x + M\mathcal{N}. \end{cases}$$

Let $n(x)$ be the unit normal vector at $X(x) = \exp_p xe_1$, $T = \frac{X'(x)}{|X'(x)|}$, $b = T \times n$, $B(x,0) \equiv 1$, $r(x,0) \equiv -\epsilon$, $s(x,0) \equiv \epsilon$, so along X

$$L = \frac{\sqrt{k}}{\epsilon},\ M = 0,\ N = -\epsilon\sqrt{k}.$$

X is a geodesic curve, and for any fixed $a \in \mathbb{R}$, $\{x = a\}$ is a geodesic curve which is perpendicular to X, so $T = V_x$ and $b = V_t$ along X. $L = \frac{\sqrt{k}}{\epsilon}$ and $B_t(x,0) = B_x(x,0) = 0 = M$, so

$$\begin{aligned} T_1(x) &= V_{xx} = \frac{\sqrt{k}}{\epsilon}\mathcal{N} \\ b'(x) &= V_{xt} = 0. \end{aligned}$$

Hence the torsion τ of X vanishes, thus X is a plane curve, and is locally convex since $\frac{\sqrt{k}}{\epsilon}\Big|_{t=0} > 0$. $\square$

Lemma 3.10 *For the isometrically immersed surface M with the metric, $ds^2 = B^2(x,t) + dt^2$ constructed in the Theorem 3.1, if ϵ is small enough, then*

$$s - r \geq c_1 \frac{B^{\frac{1}{4}}k^{\frac{1}{2}}}{t^\omega} \ \textit{for}\ t \geq T,$$

where c_1 is a constant and $0 < \omega < \frac{1}{4}$.

Proof By (3.19)

$$\partial_\beta(s-r) = \frac{s-r}{B}s_x + (s-r)\{(1-rs)\frac{B_t}{B} + \frac{1}{4}(\partial_\alpha \ln k + \partial_\beta \ln k)\}.$$

In the proof of Lemma 3.7, one has obtained that

$$s_x,\ r_x \geq -\frac{B_t}{2} \text{ for } t \geq T.$$

$s-r>0$, $\partial_x \ln k$ is bounded and $\alpha_1' t \le B \le \alpha_2' t$ for $t \ge T$, where α_1', α_2' are two constants, so by Lemma 3.7

$$\begin{aligned}\partial_\beta(s-r) &\ge (s-r)\{\frac{B_t}{2B}+\frac{1}{2}\partial_t \ln k - rs\frac{B_t}{B}+\frac{r+s}{4B}\partial_x \ln k\}\\ &\ge (s-r)\{\frac{1}{4}\partial_t \ln B+\frac{1}{2}\partial_t \ln k-\frac{\omega}{t}\} \text{ if } \epsilon \text{ is small enough,}\end{aligned}$$

where ω is a positive constant which is less than $\frac{1}{4}$. Hence

$$(s-r)(t) \ge (s-r)(T)\exp\left[\int_T^t \partial_t \ln(B^{\frac{1}{4}}k^{\frac{1}{2}}t^{-\omega})\right].$$

Thus

$$s-r \ge c_1 \frac{B^{\frac{1}{4}}k^{\frac{1}{2}}}{t^\omega}$$

where c_1 is a constant. □

Theorem 3.3 *For the isometrically immersed surface M with the metric, $ds^2 = B^2(x,t)dx^2+dt^2$ constructed in the Theorem 3.1, the mean curvature $H \to 0$ as $t \to \infty$ if ϵ is small enough.*

Proof Using that the mean curvature

$$H=\frac{1}{2}\frac{GL-2FM+EN}{EG-F^2}=\frac{1}{2}\frac{L+NB^2}{B^2},$$

and that $L=\frac{2B^2\sqrt{k}}{s-r}$, $N=\frac{2rs\sqrt{k}}{s-r}$, one obtains that

$$H=\frac{1}{2}\frac{2B^2\sqrt{k}+2rs\sqrt{k}B^2}{B^2(s-r)}=\frac{\sqrt{k}(1+rs)}{s-r}.$$

By Lemma 3.10

$$H \le \frac{(1+rs)t^\omega}{c_1 B^{\frac{1}{4}}}.$$

Using $\alpha_1' t \le B \le \alpha_2' t$ for $t \ge T$, and Lemma 3.6, one can obtain that $H \to 0$ as $t \to \infty$. □

Theorem 3.4 *Let k_1 and k_2 be the principal curvatures. For the isometrically immersed surface M with the metric, $ds^2 = B^2(x,t)dx^2 + dt^2$ constructed in the Theorem* 3.1, $|k_1 - k_2| \to 0$ *as* $t \to \infty$ *which supports the John Milnor's conjecture.*

Proof Clearly

$$|k_1 - k_2| = 2\sqrt{\frac{(k_1 + k_2)^2 - 4k_1 k_2}{4}} = 2\sqrt{H^2 - k}.$$

Since $H \to 0$ and $k \to 0$ as $t \to \infty$, $|k_1 - k_2| \to 0$ as $t \to \infty$. $\square$

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